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COMPETING LIQUIDITIES:
CORPORATE SECURITIES, REAL BONDS AND BUBBLES

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Competing Liquidities: Corporate Securities, Real Bonds and Bubbles
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ABSTRACT

We explore the link between liquidity and investment in a an overlapping generation model with a standard asynchronicity between firms' access to and need for cash. Imperfect pledgeability hinders the capacity of capital markets to resolve this asynchronicity, resulting in credit rationing and a net demand for stores of value -- liquidity -- by the corporate sector. At the heart of the model is a distinction between inside liquidity -- liquidity created within the private sector -- and outside liquidity -- assets that do not originate in private investment decisions. In the model, outside liquidity comes in two forms: rents and asset bubbles. We make four contributions. First, we show that imperfect pledgeability severs the link between dynamic efficiency and the level of the interest rate. Bubbles are possible even when the economy is dynamically efficient. Second, we demonstrate that the link between outside liquidity and investment is ambiguous: on the one hand, outside liquidity eases the asynchronicity problem of firms, boosting investment -- the liquidity effect; on the other hand it competes with inside liquidity, reduces the value of firms' collateral and lowers investment -- the competition effect. We characterize precisely the conditions under which outside liquidity and investment are complements or substitutes. Third, we explore the possibility of stochastic bubbles. We show that they trade at a liquidity discount. Bubble bursts can be endogenously triggered by bad shocks to corporate balance sheets and have potentially amplified effects on investment through liquidity dry-ups. Fourth, in an extension where corporate governance is endogenously determined by a trade-off struck by firms between collateral and value, we show that bubbles are accompanied by loose corporate governance.

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1 Introduction

Intuition and classic growth theory both suggest that stores of value and asset bubbles raise interest rates and crowd out productive investment. While the interest rate response is rather undisputed, the competition effect does not seem to capture the entire investment story in some famous episodes. Japan's bubble came with not only high interest rates but also vigorous investment and growth; when it burst, the country went through a prolonged deflation and recession. Similarly, in the US stores of values do not seem to have hampered productive investment when the public debt rose sharply during the 1980s, or during the Internet bubble; interest rates¹ and investment fell when the latter burst.

This paper provides a new and richer view on how rational bubbles impact economic activity. It builds on the idea that bubbles augment the stock of stores of value that firms can use as liquid instruments to finance their future investments. As such, bubbles are complements to productive capital.

In order to introduce a corporate demand for liquidity, its framework embodies a standard asynchronicity between firms' access to and need for cash. While this asynchronicity is perfectly resolved by capital markets in classic growth theory, capital markets are here imperfect in the tradition of corporate finance: Factors such as agency costs prevent entrepreneurs from pledging the entirety of the benefits from investment to investors, resulting in credit rationing.

More precisely, the model has overlapping generations of entrepreneurs. Entrepreneurs have some wealth when young, which they need to save for the investment opportunities that they will encounter when middle aged. Investment pays off when old (i.e., in the third period of their life). To transfer wealth between the first two periods of their life, the entrepreneurs can avail themselves of three stores of value: an exogenous flow of short term rents that produce output in the future; securities issued by previous generations of entrepreneurs' firms and therefore backed by the pledgeable income on past investments; and asset bubbles. Thus and a novel feature of our modeling, previous investment creates stores of values that new investment can build on, and so even a bubbleless economy exhibits path dependency.

We provide several examples of such "rents". In the first illustration, the state takes advantage of its regalian taxation power, and issues Treasury bonds backed by the con-

¹For example, the Fed funds rate fell from 6,5% in July 2000 to 1% in April 2004.

sumers' future income. In the second, reverse mortgages allow consumers to borrow against their future income; this securitization of their housing assets increases the number of stores of value that firms can invest in. Finally, we introduce a sector of financially unconstrained firms (i.e. firms such as depicted in standard investment theory); these firms securitize the entirety of the future income associated with their current investment. In all cases, a flow of stores of values is created by the “unconstrained sector”, that the constrained sector can build on to meet liquidity needs.

Our results can be grasped from the following insight. Firms both consume and produce stores of value. The impact of outside liquidity on investment and economic activity accordingly hinges on the relative potency of two effects: a *liquidity effect* and a *competition effect*.

On the consumption side, the firms' hoarding of liquid assets makes them benefit from an increase in the supply, and a reduction in the price of liquid assets. This investment-enhancing liquidity effect operates only when firms are financially constrained.

On the production side, their issuing securities on the capital market to finance liquidity needs makes them vulnerable to high interest rate conditions. An increase in outside liquidity raises interest rates and competes with the securities issued by the firms.

We are now equipped to enunciate and provide intuition for the main results:

Crowding in or out? Consider first the impact of a flow of rents on investment. Rents are purchased as stores of value by firms, but they also compete with the latter's security offerings. The competition effect is stronger when the investment multiplier is low. This happens when outside liquidity is scarce and when inside liquidity is neither too low nor too high. When this is the case, the competition effect dominates and rents crowd investment out. When this is not the case, the liquidity effect dominates and rents crowd investment in.

The results are sharper for bubbles. A bubble, except at its inception, always crowds investment in as long as firms are financially constrained and are net demanders of stores of value. Bubbles increase the interest rate and induce a transfer of net worth from lenders to borrowers. When the unconstrained sector is a net supplier of stores of values, bubbles crowd investment in the financially constrained sector in.²

²In a more general model where some firms are not financially constrained and some consumers undertake investments (housing, education etc.), bubbles redistribute net worth away from net suppliers of assets, and correspondingly reduce their investments while boosting the net worth and investments of other agents .

Cross-section implications. Outside liquidity impacts firms differently: Firms with limited ability to pledge future cash-flows (family and private equity firms, start ups) are not or little hit by competing claims as they issue no or few securities. Accordingly, they benefit more from a bubble, and benefit more from (or are less hurt by) an increase in the amount of rents.

Existence of bubbles. Because they are demanded as stores of value, bubbles are more likely to exist when inside and outside liquidity are scarce. Equivalently they are more likely to exist when interest rates are low.

The paper obtains three other insights.

Dynamic efficiency. Standard tests for the possibility of bubbles are ill suited for our environment. With imperfect capital markets, the economy can be dynamically efficient while the interest rate is lower than the growth rate of the economy. This is because the rate of return on internal funds exceeds that on borrowed ones; therefore the social rate of return on investments is higher than the market interest rate when returns can only be imperfectly collateralized – a result reminiscent of Saint-Paul (1992). Hence bubbles can exist even when the economy is dynamically efficient.

Bubbles and corporate governance. We study how bubbles affects the corporate governance choices of firms. If firms are confronted with a tradeoff between pledgeability and value, then states of scarce liquidity and low interest rates will lead firms to sacrifice value in order to boost collateral. Bubbles increase liquidity, lower the equity multiplier and lead firms to loosen corporate governance by forgoing pledgeability for value.

Bubbly liquidity discount. We examine the possibility of stochastic bubbles – bubbles that can burst. Bubble bursts are accompanied by low interest rates and high equity multipliers. Because stochastic bubbles pay off only in states of the world where equity multipliers are low, they command a liquidity discount – they have higher expected returns. We show that bad shocks hitting firms’ balance sheets reduce the demand for liquidity and lead endogenously to bubble bursts. Bad shocks to corporate balance sheets can potentially have an amplified effect on investment over and above that described in the literature emphasizing the importance of corporate net worth – for example Kiyotaki-Moore (1997) – by triggering liquidity dry-ups in the form of bubble bursts.

Outline of the paper. The paper proceed as follows. Section 2 sets up the model and describes the solution when there are no bubbles. It characterizes its unique steady state

and derives some key comparative statics results. Section 3 introduces the possibility of rational asset price bubbles. It explicits the dynamics with bubbles, describes the properties of the unique bubbly steady state and analyzes how bubbles affect corporate governance choices of firms. Section 4 introduces stochastic bubbles and derives the mechanics of a bubbly boom-bust episode. Section 5 checks the robustness of the results in several variants of the model, and Section 6 summarizes the main insights and discusses alleys for research.

Relationship to the literature

The paper builds on a number of contributions. Most obviously, it brings together the literature on (rational) bubbles and that on aggregate liquidity, hence its title. The competition effect dates back to at least Diamond (1965)'s celebrated analysis of national debt, and is prominent in the theory of rational bubbles (Tirole 1985). Indeed, the two standard criticisms of the latter theory are that it predicts a crowding out of investment by bubbles ("crowding-out critique") and that bubbles can exist only if the productive sector consumes more resources than it delivers (i.e., only if the economy is dynamically inefficient), which is empirically debatable (Abel et al. 1989's "dynamic efficiency critique"). This paper shows that these two concerns disappear under imperfect capital markets.

The role of stores of values in supporting investment when income is not fully pledgeable has been stressed for example by Woodford (1990) and Holmström-Tirole (1998). In Woodford's contribution, firms are net lenders and there is always a need for (and a potential shortage of) stores of value. Woodford assumes away the competition effect by positing that none of the future cash flow is pledgeable to investors and so firms do not issue securities. By contrast, firms in Holmström-Tirole are net borrowers, and shortages of liquidity are associated with adverse macroeconomic shocks. Holmström and Tirole also assume away the competition effect, but for a different reason: In their model, security issues never compete with liquidity that issuing firms have no use for, unlike in this paper. This paper takes the Woodford approach for illustrative purposes. Saint-Paul (2005) shows that government debt (a store of value), while deterring capital accumulation, can increase the efficiency of the financial sector. Entrepreneurs can buy public debt and use it as collateral. The existence of collateral reduces agency costs (Saint-Paul uses the costly-state-verification model as an illustration). Accordingly, public debt boosts growth over a range of parameters.

The paper shares with Kiyotaki-Moore (1997) the idea that investment decisions are intertemporal complements. In Kiyotaki-Moore, *tomorrow's* investment will raise the price of the store of value, which is used as an input in the production process; this future increase in the price of the store of value raises the firms' wealth and thereby today's investment. In our paper, it is *yesterday's* investment that supports today's investment, by creating securities that firms can hoard to meet their liquidity needs. Also, Kiyotaki-Moore's focus is rather different as it has no bubbles and does not emphasize the efficiency test.

The rational bubble literature has addressed the crowding-out critique in alternative ways. Bubbles are attached to investment in Olivier (2000) and to entrepreneurship in Ventura (2003), generating an incentive and a wealth effect respectively; in both papers, bubbles can crowd investment in. Saint-Paul (1992) addresses the dynamic-efficiency critique by studying an endogenous growth model with bubbles, in which the social return on investment exceeds the private return due to spillovers. The long-term rate of interest can then be smaller than the rate of growth of the economy, and yet the economy be dynamically efficient. Caballero and Krishnamurthy (2006) develop a theory of bubbles in emerging markets. They introduce, as we do, an investment driven demand for liquidity and show in the presence of fragile (stochastic) bubbles, the economy overinvests in the bubbly asset and is overexposed to bubble crashes due to a pecuniary externality.

Our paper sheds some light on the debate as to whether monetary authorities should try to lean against bubbles (or, in a more extreme form, try to make them pop) by raising interest rates or denying access to the discount window to banks that extend too many loans. Some scholars (Bernanke 2002, Bernanke-Gertler 2000, 2001, Gilchrist-Leahy 2002) argue that the central bank should not pay attention to asset prices unless these signal future inflation; others (Bordo-Jeanne 2002) are in favor of a moderate reaction. All concur that a restrictive policy leads to a lower output and a significant risk of collateral — induced — credit crunch. Our model is consistent with this premise, as the pricking of the bubble leads to a collateral shortage and reduced investment and production.

2 The Bubbleless Economy

2.1 Model set-up

Consider a single good overlapping generation model with a growing population of risk-neutral entrepreneurs and consumers. The population growth rate is $1 + n$. Entrepreneurs live for three periods: young, middle age and old. Consumers live for only two periods: young and old.

In addition to investment projects carried out by entrepreneurs, there are l unit of rents in period t . Rents at date t are short-term real bonds, paying one unit of good at date $t + 1$. Let r_t denote the interest rate between dates t and $t + 1$.

When young, entrepreneurs are endowed with A units of good (wealth) and $(1 - \theta) l$ unit of rents per capita. When middle aged, they can invest i_t to get $\rho_1 i_t$ when old. However, only a fraction $\rho_0 i_t < \rho_1 i_t$ of the return on investment is pledgeable. In equilibrium, it will always be the case that $\rho_0 < 1 + r_{t+1}$ so that firms can only partially rely on outside financing at the investment stage. Throughout the paper, we consider only the regime where entrepreneurs are constrained in their investment. In Section 5.1 we relax this assumption and introduce an unconstrained corporate sector.

Assumption 1 (*financially constrained regime*) $\rho_1 > 1 + r_{t+1}$ and $1 + n > \rho_0$.³

The rest of the rents θl is owned by consumers who are not entrepreneurs. Until Section 3.4, we assume that $\theta \in [0, 1]$. In Section 3.4, we allow for $\theta < 0$ and analyze in detail the consequences of this important assumption.

In their youth, entrepreneurs of generation t must decide how much additional bonds \hat{l}_t to purchase, and how much to invest x_t in projects of entrepreneurs of generation $t - 1$ realized in period t and delivering output in period $t + 1$.

To begin with, we assume that entrepreneurs can only consume when old, and that consumers can only consume when young. Later in Section 5 we will allow for less extreme preferences: linear or concave utilities with per period discount factor β . We thereby ignore in a first step the possibility that consumers save part of their endowment and ensure that entrepreneurs save theirs and invest it in productive assets. Consumers of generation t therefore sell their rents θl to the entrepreneurs of generation t .

³In a linear model such as ours, when outside liquidity is too abundant, the interest rate $1 + r_{t+1}$ rises above ρ_1 and entrepreneurs abandon their productive investment projects. Instead they hoard outside liquidity in order to finance consumption in the last period of their life.

Entrepreneurs invest all their wealth in their youth in stores of values – rents and investment projects of the previous generation – and use these savings when middle-aged to produce collateral for their investment project:

$$A = \frac{\hat{l}_t}{1 + r_t} + x_t.$$

At date $t + 1$, the date- t agent's borrowing capacity is the sum of the value of claims on future income, $\rho_0 i_t / [1 + r_{t+1}]$, the yield on hoarded rents $\hat{l}_t + (1 - \theta)l$, and the return, $(1 + r_t)x_t$, on securities purchased from the previous generation:

$$i_t = \frac{\rho_0 i_t}{1 + r_{t+1}} + [\hat{l}_t + (1 - \theta)l] + (1 + r_t)x_t.$$

Market clearing therefore requires

$$\hat{l}_t = \theta l \text{ and } x_t = \frac{\rho_0 i_{t-1}}{(1 + n)(1 + r_t)}.$$

2.2 Competitive equilibrium

The economy can be described recursively with one state variable: past investment i_{t-1} . At date $t + 1$, given past investment i_{t-1} , current investment i_t and the interest rate r_{t+1} are jointly determined by the intersection of a supply and a demand equation for assets. That these two curves intersect is the condition for the market of stores of value to clear at date $t + 1$. The demand equation is independent of i_{t-1} . The supply equation on the other hand depends on past investment i_{t-1} , which determines the liquidity available for current investment. Hence liquidity imparts a path dependency to the economy.

Asset supply equation. The supply equation describes how generation t 's investment at date $t + 1$ is constrained by the available liquidity, $l + \frac{\rho_0 i_{t-1}}{1+n}$, and by the investment-related pledgeable income, $\frac{\rho_0 i_t}{1+r_{t+1}}$:

$$i_t = \frac{\rho_0 i_t}{1 + r_{t+1}} + l + \frac{\rho_0 i_{t-1}}{1 + n} \tag{1}$$

and can be expressed as

$$i_t = i^s(i_{t-1}, r_{t+1}) \equiv \frac{l + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \frac{\rho_0}{1+r_{t+1}}}$$

with $\partial i^s(i_{t-1}, r_{t+1})/\partial r_{t+1} < 0$.

Asset demand equation. The demand equation says that generation $t+1$'s wealth goes into buying liquidity from the consumers (θl) and that generated by the previous generation's investment ($\rho_0 i_t/(1+n)$):

$$A(1+r_{t+1}) = \theta l + \rho_0 \frac{i_t}{1+n} \quad (2)$$

It can be expressed as

$$i_t = i^d(r_{t+1}) \equiv \frac{1+n}{\rho_0} [A(1+r_{t+1}) - \theta l]$$

with $\partial i^d(r_{t+1})/\partial r_{t+1} > 0$.

We define a competitive equilibrium as a sequence of investment levels and interest rates $\{i_t, r_t\}$ such that at every date t , the asset market clears:

Definition 1 *A competitive equilibrium is a sequence $\{i_t, r_t\}_{t \geq 0}$ together with an initial investment level $i_{-1} \geq 0$ such that the asset supply and asset demand equations (1) and (2) hold and for all $t \geq 0$, $i_t \geq 0$ and $1+r_t > 0$.*

The asset market clears at date $t+1$ when the demand and the supply curves intersect, determining i_t and r_{t+1} as a function of the state variable i_{t-1} . This involves solving a quadratic equation:

$$\left[1 - \frac{\rho_0 A}{\theta l + \frac{\rho_0 i_t}{1+n}}\right] i_t = l + \rho_0 \frac{i_{t-1}}{1+n} \quad (3)$$

We derive the exact solution for investment dynamics – i_t as a function of i_{t-1} – in appendix A1.⁴

⁴A simple case is $\theta = 0$. In that case, the system can be solved in closed form:

$$\begin{aligned} 1+r_{t+1} &= \frac{\rho_0 \left(A(1+n) + l + \rho_0 \frac{i_{t-1}}{1+n} \right)}{A(1+n)} \\ i_t &= A(1+n) + l + \frac{\rho_0}{1+n} i_{t-1} \end{aligned}$$

Conditions (1) and (2) imply that the productive sector provides its own liquidity in a dynamic fashion: an increase in i_{t-1} , using (1), leads to an increase in i_t (and in r_{t+1} , which from (2), must then co-vary with i_t).

An increase in the pledgeability parameter ρ_0 shifts the asset demand curve upwards and the asset supply curve downwards. This increases the interest rate and has an ambiguous effect on investment i_t : we can only say for sure that total pledgeable income $\rho_0 i_t$ increases. Note also that as is standard from the corporate finance literature, investment increases with the entrepreneurs' wealth A : higher net worth pushes the demand curve upwards, decreasing the interest rate r_{t+1} and increasing investment i_t .

By contrast, the rate of growth of the economy, n , impacts investment in two opposite ways. On the one hand, the current generation builds on a smaller amount of per-capita liquidity provided by the previous generation. On the other hand, the collateral that the current generation will create will be more valuable as the next generation's savings will be abundant and will be used to purchase stores of value. Interestingly, none of these effects would exist in a neo-classical model; here the marginal productivity of investment, ρ_1 , is constant and so in the absence of credit constraints, per-capita investment would not depend on the rate of growth of the economy.

The asset supply and asset demand equations (1) and (2) can also be used to determine the impact how outside liquidity – that is, rents l – impacts investment. The impact of an increase in the level of rents l can be decomposed into two effects. On the one hand, increasing rents available at date t shifts the asset supply curve (1) upwards, raising investment i_t for all interest rate levels r_{t+1} – a *liquidity effect*. On the other hand, increasing rents available at date $t+1$ shifts the asset demand curve downwards, lowering investment i_t for all interest rate levels r_{t+1} – a *competition effect*. The interest rate r_{t+1} unambiguously increases, but the resulting effect on investment i_t at date $t+1$ is ambiguous. Firms demand liquidity which is akin to an input in production. This tends to make investment and outside liquidity complements. But investments made by the private sector also play the role of inside liquidity. Inside liquidity is in direct competition with outside liquidity. This tends to make investment and outside liquidity complements. This distinction between the liquidity effect and the competition effect also has a temporal dimension. Past liquidity – inside liquidity i_{t-1} or outside liquidity – and contemporaneous investment i_t are complements. Future liquidity and contemporaneous investment i_t are

The system is stable since $\frac{\rho_0}{1+n} \leq 1$.

substitutes.

2.3 Steady states

The basic model has a unique steady state determined by the unique intersection with $i^* > 0$ of the steady-state asset supply and demand curves:

$$i^* = \frac{l}{1 - \frac{\rho_0}{1+r^*} - \frac{\rho_0}{1+n}} \text{ and } i^* = \frac{1+n}{\rho_0} [A(1+r^*) - \theta l]$$

In appendix A1, we solve for i^* and r^* in closed form.⁵ As will become clear when we compute the dynamics for investment in section 3, this equilibrium is stable.

Increasing outside liquidity (l) or collateral (ρ_0) shifts the supply curve upwards and the demand curve downwards. Therefore, the interest rate r^* increases with l and ρ_0 . How investment i^* varies with outside liquidity l and collateral ρ_0 however, is a priori ambiguous. More rents, or more collateral are good news for investors demanding liquidity – the liquidity effect–, but bad news for those supplying it– the competition effect– as it introduces competition for their stores of values, depresses their price, and hence reduces their net worth. When $\theta < 1$, there is also a *wealth effect* since increasing l increases the net worth of entrepreneurs, which increases investment. In other words whether outside liquidity or rents are complements or substitutes with investment is ambiguous and depends on whether the liquidity effect and the wealth effect are stronger or weaker than the competition effect.

2.4 Rents and investment

In this section, we find conditions under which outside liquidity increases investment. In the model the amount of rents l parametrizes the level of outside liquidity in the economy. It is useful to flesh out the concept of liquidity however. In our view, l can typically come from consumer leverage and securitization or be provided by the state in the form of public debt as in Woodford (1990).

⁵Note that when l is equal to 0, we get the remarkably simple expression

$$i^*|_{l=0} = \frac{A(1+n)}{1 - \frac{\rho_0}{1+n}} \text{ and } 1+r^*|_{l=0} = \frac{\rho_0}{1 - \frac{\rho_0}{1+n}}$$

2.4.1 What rents are: some microfoundations

Public supply of liquidity. A first microfoundation for rents l goes as follows (state provided liquidity): consumers live for one period, receive income w at home or abroad. They incur a cost $\tilde{l} < w$ if they move abroad. So the state can tax them \tilde{l} . The state issues bonds one period ahead. Let π be the number of newly born consumers per newly born entrepreneur and define $l \equiv \tilde{l}\pi(1+n)$. The state receives $l/(1+r_t)$ from the bond issuance and distributes it to consumers and firms in proportion $(\theta, 1-\theta)$. Note that individual consumers live for a single period. Individually, they are neither lenders nor borrowers. Collectively, though, they are net borrowers as the state issues "on their behalf" pledges on their future income.

A private-sector variant of this would have private lenders, who subsidize consumption when young of two-period lived consumers up to a reimbursement limit of l as consumers can move abroad in the second period of their life. This model is isomorphic to the one with public supply just outlined with the additional constraint that $\theta = 1$.

Securitization. Alternatively, we could suppose that consumers have some endowment of goods w – labor income – in their youth. They use that labor income to build a house, which has total value $y_1 j_t$ at period $t+1$, where j_t is the home investment realized in period t . The house might have some private value on top of its rental value. Suppose first that only the rental value $y_0 j_t < y_1 j_t$ can be securitized today. Consumers can invest up to $\frac{w}{1+r_t}$ in housing. Consumers thus create $l_t = \frac{y_0 w}{1+r_{t+1}}$ additional stores of values for the corporate sector. In that model, we implicitly have $\theta = 1$. An increase in securitization – in the form of mortgage backed securities for example – can be formalized as an increase in y_0 towards y_1 and materializes as an increase in l_t . A feature of this microfoundation is that the amount of rents l_t is endogenous as it is affected by the level of interest rates (we generalize the analysis to interest-dependent rents in Section 5.1).

Consumers as borrowers. We will also analyze a less extreme case where consumers have concave preferences and hence an elastic borrowing margin. They live for two periods and have preferences given by

$$u(c_y) + \beta u(c_o)$$

where c_y and c_o denote respectively consumption when young and old. They earn income w_y when young and w_o when old. To simplify the analysis, we focus on the case of log preferences where $u(c) = \log(c)$. In this case, consumers of generation t facing interest

rate r_t consume

$$c_{y,t} = \frac{1}{1 + \beta} \left(w_y + \frac{w_o}{1 + r_t} \right) \text{ and } c_{o,t} = \frac{\beta}{1 + \beta} ((1 + r_t)w_y + w_o)$$

The supply of rents from the consumers' sector is therefore

$$l_t = l(r_t) \equiv w_o - \frac{\beta}{1 + \beta} ((1 + r_t)w_y + w_o)$$

where $l(r_t)$ is decreasing with r_t . We analyze this setup in greater detail in Section 5.1.

Unconstrained firms. Suppose that there also exists a competitive fringe of firms operating a concave production function $f(k_t)$. These firms are owned by consumers who only consume when young. Consumers then sell the firms to investors for a price $f(k_t)/(1 + r_t) - k_t$ where k_t is the equilibrium investment level. In equilibrium, it will be the case that $f'(k_t) = 1 + r_t$ so that $k_t = k(r_t)$ where k is decreasing in r_t . The model is then nested by the one described in the above paragraph with $l(r_t) = f(k(r_t))$.

2.4.2 The horserace between the liquidity effect, the wealth effect and the competition effect

Let us clarify under which circumstances outside liquidity (rents) and investment are complements or substitutes in steady states. Note first that if Assumption 1 does not hold, then our characterization of the steady state is invalid. This happens when liquidity l is so high that the interest rate $1 + r_{t+1}$ exceeds the rate of return on productive projects ρ_1 . Entrepreneurs then give up entirely on their investment projects and instead hoard outside liquidity to finance consumption when old. Investment is completely crowded out and liquidity is not valued.

In this paper, we are chiefly interested in the regime where liquidity is scarce and Assumption 1 holds. We start with a simple case, $\theta = 1$, which has the virtue of neutralizing the wealth effect of rents on the net worth of entrepreneurs. We have in that case

$$i^* = \frac{A(1 + r^*)}{1 - \frac{\rho_0}{1 + r^*}}$$

This expression shows that i^* increases in r^* and therefore in l if and only if

$$1 > \frac{2\rho_0}{1+r^*} \quad (4)$$

and decreases otherwise. Moreover, one can show that $\frac{2\rho_0}{1+r^*}$ is non-monotonic in ρ_0 : increasing then decreasing. In addition, if $\frac{\rho_0}{1+r} \geq \frac{1}{2}$, then (4) is automatically verified for all $l \geq 0$.

Hence there are generally two regions (one of them might not exist). For l low enough, r^* is low and condition (4) is violated: in this region i^* decreases with l . For l high enough, r^* is high and condition (4) holds: in this region, i^* increases with l . This suggests that l and i^* are substitutes at low levels of stores of values and complements at higher levels of stores of values. When outside liquidity is scarce, the *competition effect* dominates and investment decreases with liquidity. By contrast, when outside liquidity is abundant, the *liquidity effect* dominates and investment increases with liquidity.

This can be understood as follows. Increasing l increases liquidity and therefore decreases the price of liquidity by increasing the interest rate r^* . A higher interest rate r^* on the one hand increases the return on savings and therefore the total net worth of entrepreneurs at the date of investment $A(1+r^*)$, and on the other hand reduces the price of collateral and therefore the investment multiplier $(1 - \frac{\rho_0}{1+r^*})^{-1}$. The latter effect is stronger when $\frac{\rho_0}{1+r^*}$ is high. When l is high then the interest rate is high, $\frac{\rho_0}{1+r^*}$ is low and the former effect dominates.

It can be shown that $\frac{\rho_0}{1+r^*}$ is non monotonic in ρ_0 : first increasing and then decreasing. The investment multiplier is not very sensitive to the interest rate r^* both for low and high values of ρ_0 . Therefore, (4) is more likely to be violated for intermediate values of ρ_0 . When there is no inside liquidity ($\rho_0 = 0$), then there is no collateral, the investment multiplier is equal to 1 and the latter effect vanishes so that investment i^* increases with outside liquidity (l). When inside liquidity is high enough ($\rho_0 \geq \frac{1}{2}$), then $\frac{\rho_0}{1+r^*}$ is low and the investment multiplier does not vary much with l . As a result, investment i^* also increases with outside liquidity. We generalize those insights in Proposition 1 below.

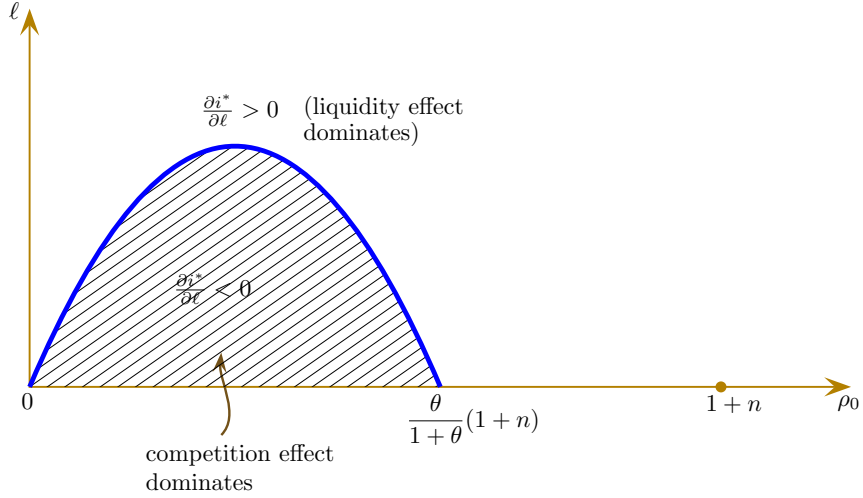


Figure 1

Summing up:

Proposition 1 *In the bubbleless economy, steady state per capita investment i^**

(i) *grows with the fraction of rents owned by the entrepreneurs $(1 - \theta)$,*

(ii) *may increase or decrease with the rate of growth of the economy (n) and with pledgeable income (ρ_0),*

(iii) *when inside liquidity is plentiful ($\frac{\rho_0}{1+n} \geq \frac{\theta}{1+\theta}$), and so when interest rates are high, grows with outside liquidity (l),*

(iv) *when inside liquidity is scarce ($\frac{\rho_0}{1+n} \leq \frac{\theta}{1+\theta}$), grows (decreases) with outside liquidity (l) when inside liquidity is plentiful (scarce), and so when interest rates are high (low).*

More precisely, there exists $l_0(\frac{\rho_0}{1+n}, A, 1+n, \theta) > 0$ such that $\frac{\partial i^}{\partial l} < 0$ for $l \in [0, l_0)$ and $\frac{\partial i^*}{\partial l} > 0$ for $l \in (l_0, +\infty)$ ⁶. Moreover, l_0 is non-monotonic in $\frac{\rho_0}{1+n}$: $l_0(0, A, 1+n, \theta) = l_0(\frac{\theta}{1+\theta}, A, 1+n, \theta) = 0$.*

The intuitions behind (iii) and (iv) are very similar to the ones we developed for the case $\theta = 1$. This can be understood most clearly by noting that

$$i^* = \frac{A(1+r^*) + (1-\theta)l}{1 - \frac{\rho_0}{1+r^*}}.$$

⁶The exact expression for l_0 is

$$l_0\left(\frac{\rho_0}{1+n}, A, 1+n, \theta\right) \equiv \frac{2A(1+n)\frac{\rho_0}{1+n}\left(1 - \frac{\rho_0}{1+n}\frac{1+\theta}{\theta}\right)}{\theta\left(1 + \frac{\rho_0}{1+n}\frac{1-\theta}{\theta}\right)^2}$$

Note also that inside liquidity $\rho_0 i^*$ always increases with pledgeable income ρ_0 . It might be the case, however, that i^* decreases with ρ_0 : on the one hand, for a given level of demand for liquidity $A(1 + r^*) = \frac{\rho_0}{1+n} i^* + \theta l$, a higher level of pledgeable income ρ_0 decreases investment i^* ; on the other hand, r^* increases with ρ_0 and the net effect is ambiguous.

3 Bubbles

Let us now allow for the possibility of bubbles.

3.1 Competitive equilibrium and steady state

Let b_t be the size of the bubble per capita. By convention, b_t is the bubble at date $t + 1$ per entrepreneur of generation t . Bubbles affect both the asset supply and the asset demand equations. We modify our definition of a competitive equilibrium accordingly.

Definition 2 *A competitive equilibrium is a sequence $\{i_t, b_t, r_t\}_{t \geq 0}$ together with an initial investment level $i_{-1} \geq 0$ such that the asset supply and asset demand equations (5) and (6) defined below hold and for all $t \geq 0$, $i_t \geq 0$, $b_t \geq 0$ and $1 + r_t > 0$.*

The economy can now be described recursively with two state variables: i_{t-1} and b_t . The supply equation becomes

$$i_t = i^s(i_{t-1}, r_{t+1}; b_t) \equiv \frac{b_t + l + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \frac{\rho_0}{1+r_{t+1}}} \quad (5)$$

with $\partial i^s(i_{t-1}, r_{t+1}; b_t) / \partial r_{t+1} < 0$.

The demand equation becomes

$$i_t = i^d(r_{t+1}; b_t) \equiv \frac{1+n}{\rho_0} [A(1+r_{t+1}) - \theta l - \frac{1+r_{t+1}}{1+n} b_t], \quad (6)$$

with $\partial i^d(r_{t+1}; b_t) / \partial r_{t+1} > 0$.

The bubble shifts the supply curve up and the demand curve down. Therefore it unambiguously increases the interest rate, but has an *a priori* ambiguous effect on investment. In Proposition 2, we will resolve this ambiguity and show that bubbles increase current and future investment.

Bubbly steady state. There exists either zero or a unique steady state with bubbles. When it exists it is given by

$$\begin{aligned} i^{**} &= \frac{(1 - \theta)l + A(1 + n)}{1 - \frac{\rho_0}{1+n}} \\ b^{**} &= [(1 - \theta)l + A(1 + n)] \frac{1 - 2\frac{\rho_0}{1+n}}{1 - \frac{\rho_0}{1+n}} - l \\ r^{**} &= n \end{aligned}$$

Let

$$\Lambda \equiv \frac{l}{(1 - \theta)l + A(1 + n)}$$

denote the ratio of outside liquidity over corporate wealth, or “outside liquidity ratio” for short. The condition of existence of a bubbly steady state is

$$\frac{1 - 2\frac{\rho_0}{1+n}}{1 - \frac{\rho_0}{1+n}} > \Lambda \tag{B}$$

Condition (B) shows that bubbles can emerge when inside (ρ_0) and outside (Λ) liquidity are scarce, creating a high demand for stores of value. Note also that in a bubbly steady state, the interest rate is pinned down at n . The analysis of the phase diagram below shows that condition (B) is equivalent to the standard condition that the interest rate in the bubbleless steady state r^* be less than n .

When $\theta = 1$, variations in l are compensated one for one with variations in the size of the bubble: the number of stores of values is invariant to l . When $\theta < 1$ on the other hand, rents have a positive wealth effect on entrepreneurs and as a result, investment increases with l . The bubble only partially crowds out rents, and the number of stores of value increases with l .

Investment dynamics. One can eliminate the rate of interest and rewrite generation- t 's investment as a function of the previous generation's investment and the bubble:

$$i_t = (1 + n)A + \frac{\rho_0 i_{t-1}}{1 + n} + \left[1 - \frac{\theta(1 + n)}{\rho_0}\right]l + \frac{\theta(1 + n)l}{\rho_0 i_t} \left[b_t + l + \frac{\rho_0 i_{t-1}}{1 + n}\right].$$

Lemma 1 *Investment i_t is an increasing function of i_{t-1} and b_t .*

The economy is a two-dimensional dynamic system that can be described conveniently with a phase diagram. This requires characterizing $\dot{i}_t = \dot{i}_{t-1}$ schedule and the $\mathbf{b}_{t+1} = \mathbf{b}_t$ schedule. The $\dot{i}_t = \dot{i}_{t-1}$ schedule is given by

$$\mathbf{b}_t = \dot{i}_{t-1}^2 \frac{\rho_0}{\theta \mathbf{l}(1+n)} \left(1 - \frac{\rho_0}{1+n} \right) - \frac{\rho_0 \dot{i}_{t-1}}{\theta \mathbf{l}(1+n)} [(1+n)\mathbf{A} + (1-\theta)\mathbf{l} + [2 - \frac{(1+n)}{\rho_0}]\theta \mathbf{l}] - \mathbf{l}.$$

This defines the schedule as a function \mathbf{b}_t of \dot{i}_{t-1} : $\mathbf{b}_t^i(\dot{i}_{t-1})$.

The $\mathbf{b}_{t+1} = \mathbf{b}_t$ schedule is given by

$$\mathbf{b}_t = - \left(\frac{\rho_0}{1+n} \right)^2 \dot{i}_{t-1} + \left(1 - \frac{\rho_0}{1+n} \right) [\mathbf{A}(1+n) + (1-\theta)\mathbf{l}] - \mathbf{l}$$

which defines a schedule $\mathbf{b}_t^b(\dot{i}_{t-1})$.

We have $\mathbf{b}_t^i(0) = -\mathbf{l}$ and $\mathbf{b}_t^b(0) = \left(1 - \frac{\rho_0}{1+n} \right) [\mathbf{A}(1+n) + (1-\theta)\mathbf{l}] - \mathbf{l}$ which is strictly positive as long as (B) holds. It is easy to verify that \mathbf{b}_t^i is increasing when it intersects \mathbf{b}_t^b . The sign of $\frac{d\mathbf{b}_t^i}{d\dot{i}_{t-1}}|_{\dot{i}_{t-1}=0}$ on the other hand, is unclear a priori. Note that the bubbleless steady state is always stable.

When $\theta \mathbf{l} = 0$, the bubble has no impact on investment. The bubble just increases the interest rate but does not have any impact on the dynamics of investment. When $\theta \mathbf{l} > 0$ on the other hand, the bubble increases investment along the path to the bubbly steady state and at the steady state.

While the bubble, like the rents, acts as a store of value, it does not have their ambiguous impact on investment. To understand this, consider, first, the direct effect of a bubble, a 1-for-1 crowding out of the value of securities issued by entrepreneurs: the total source of liquidity for entrepreneurs is $\mathbf{b}_t + \mathbf{l} + \frac{\rho_0 \dot{i}_t}{1+r_{t+1}}$, of which the bubble \mathbf{b}_t and the securities $\frac{\rho_0 \dot{i}_t}{1+r_{t+1}}$ are sold to the next generation of entrepreneurs. Put differently, only the sum $\mathbf{b}_t + \frac{\rho_0 \dot{i}_t}{1+r_{t+1}}$ enters the supply and demand equations. The presence of the bubble therefore increases the rate of interest. This increase in the interest rate lowers the value of the rents. Thus when entrepreneurs buy rents from consumers ($\theta > 0$), in the competition for savings between the two sources of liquidity owned by entrepreneurs (bubble, securities to be issued) and the one held by consumers (rents), the increase in interest rate benefits the liquidity held by entrepreneurs and therefore crowds investment in.

Figure 2 is a phase diagram representing the dynamics of the economy. Note that the bubbly steady state always features more investment than the bubbleless steady state. It features a downward sloping saddle path. If the economy starts on the saddle path, it will eventually converge to the bubbly steady state. If it starts below the saddle path, it will eventually converge to the bubbleless steady state. The economy cannot start above the saddle path without eventually violating one of the constraints.

We are now in position to describe the dynamics when a bubble pops up. Suppose for example that we are in the steady state without bubbles. As the bubble pops up, the economy jumps upwards to reach the saddle path of the bubbly steady state. Investment booms, the interest rate increases and the bubble gradually decreases as the economy converges to the bubbly steady state. More generally, the following proposition summarizes the effects of a bubble:

Proposition 2 *Assume that (B) holds. For any i_{t_0-1} , there exists a maximum feasible bubble $\bar{b}(i_{t_0-1})$. The path of productions/investments $\{i_t\}_{t \geq t_0}$ and interest rates $\{r_t\}_{t \geq t_0}$ are increasing in the size of the original bubble b_{t_0} . For $b_{t_0} < \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbleless: it converges to the bubbleless steady state. For $b_{t_0} = \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbly: it converges to the bubbly steady state.*

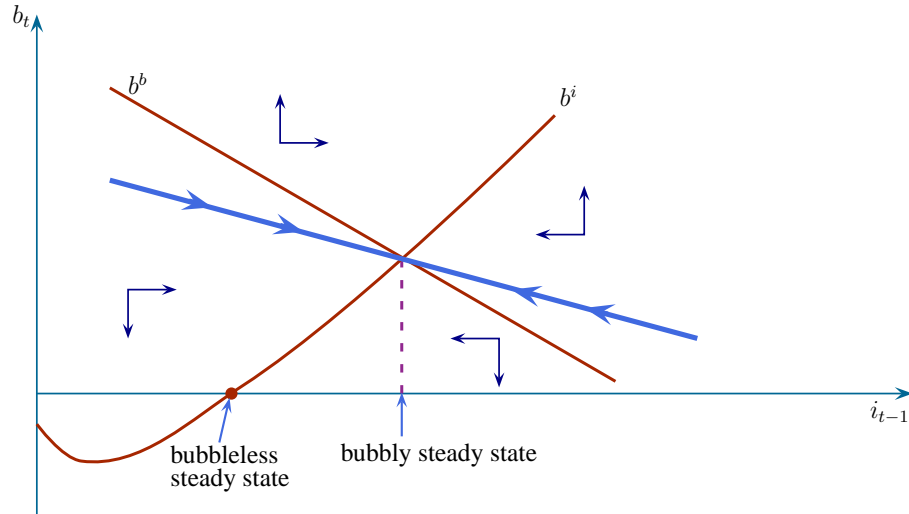


Figure 2: Phase diagram when consumer sector is a net borrower

Corollary 1 *The condition for a bubble to exist (B) is equivalent to $r^* < n$.*

Proof. Note that (B) amounts to saying that \mathbf{b}^b intersects \mathbf{b}^i at a point where $\mathbf{b}_t > 0$. Note further that the interest rate at the intersection of the two schedules is always equal to \mathbf{n} . Since the interest rate is increasing in \mathbf{i}_{t-1} and \mathbf{b}_t , \mathbf{r}^* is less than \mathbf{n} if and only if (B) holds. ■

Proposition 3 *Assume that (B) holds. On an asymptotically bubbly path, the bubble*
(i) decreases with the fraction of income that is pledgeable (ρ_0); indeed a bubble can exist if and only if the pledgeable income is smaller than a threshold;
(ii) decreases with the number of existing stores of value (\mathbf{l}).

3.2 Collateral heterogeneity

We have assumed for convenience that firms are homogenous (perhaps up to a scaling factor). When firms differ, say, in the pledgeability of their income, those with limited access to unsophisticated investors, i.e., low ρ_0 firms (family firms, private equity, startups), benefit relatively more from the presence of a bubble: They enjoy the liquidity effect without being much impacted by the competition effect as they do not resort much to small investors' money.

In fact, let \mathbf{k} be an index for firms and let ρ_0^k be an increasing function of \mathbf{k} . We can assume without loss of generality that \mathbf{k} is distributed uniformly on $[0, 1]$. We then have the following aggregation result. The economy is described by two state variables: the value of the bubble \mathbf{b}_t and the integral $\int \rho_0^k \mathbf{i}_{t-1}^k d\mathbf{k}$. The law of motion for \mathbf{b}_t is still $\mathbf{b}_{t+1} = \frac{\mathbf{b}_t(1+\mathbf{r}_{t+1})}{1+\mathbf{n}}$, while $\int \rho_0^k \mathbf{i}_{t-1}^k d\mathbf{k}$ and \mathbf{r}_{t+1} are jointly determined as the intersection of the aggregate supply and the demand curves for assets:

$$\int \rho_0^k \mathbf{i}_t^k d\mathbf{k} = \left(\int \frac{\rho_0^k}{1 - \frac{\rho_0^k}{1+\mathbf{r}_{t+1}}} d\mathbf{k} \right) \left(\mathbf{b}_t + \mathbf{l} + \frac{\int \rho_0^k \mathbf{i}_{t-1}^k d\mathbf{k}}{1 + \mathbf{n}} \right)$$

and

$$\int \rho_0^k \mathbf{i}_t^k d\mathbf{k} = (1 + \mathbf{n}) \left[A (1 + \mathbf{r}_{t+1}) - \theta \mathbf{l} - \frac{1 + \mathbf{r}_{t+1}}{1 + \mathbf{n}} \mathbf{b}_t \right].$$

Investment by firm k can be computed as

$$i_t^k = \frac{b_t + l + \frac{\int \rho_0^k i_{t-1}^k dk}{1+n}}{1 - \frac{\rho_0^k}{1+r_{t+1}}}.$$

There exists either zero or a unique steady state with bubbles. When it exists it is given by

$$\begin{aligned} i^{**k} &= \frac{A(1+n) + (1-\theta)l}{1 - \frac{\rho_0^k}{1+n}} \\ b^{**} &= \left(\int \frac{1 - 2\frac{\rho_0^k}{1+n}}{1 - \frac{\rho_0^k}{1+n}} dk \right) [A(1+n) + (1-\theta)l] - l \\ r^{**} &= n. \end{aligned}$$

The condition for a bubble to exist is now given by

$$\left(\int \frac{1 - 2\frac{\rho_0^k}{1+n}}{1 - \frac{\rho_0^k}{1+n}} dk \right) > \Lambda. \quad (B^k)$$

The analysis of the dynamics of the economy are exactly as in Section 3.1.

Replacing the representative firm's pledged income by the industry-average pledged income, we see that the previous analysis generalizes to heterogenous firms. Hence the investment i_t^k of firms with lower pledgeable income ρ_0^k increases relatively more with the size of the bubble. Indeed

$$\begin{aligned} \frac{di_t^k}{i_t^k db_t} &= \frac{\partial i_t^k}{i_t^k \partial b_t} + \frac{\partial i_t^k}{i_t^k \partial r_{t+1}} \frac{dr_{t+1}}{db_t} \\ &= \frac{1}{b_t + l + \frac{\int \rho_0^k i_{t-1}^k dk}{1+n}} - \frac{\rho_0^k}{(1+r_{t+1} - \rho_0^k)^2} \frac{dr_{t+1}}{db_t} \end{aligned}$$

is decreasing in k .

Proposition 4 *Assume that (B^k) holds. Then:*

(i) *for any value of $\int \rho_0^k i_{t_0-1}^k dk$, there exists a maximum feasible bubble \bar{b} ($\int \rho_0^k i_{t_0-1}^k dk$).*

The path of productions/investments $\{i_t\}_{t \geq t_0}$ and interest rates $\{r_t\}_{t \geq t_0}$ are increasing in the size of the original bubble b_{t_0} . For $b_{t_0} < \bar{b} \left(\int \rho_0^k i_{t_0-1}^k dk \right)$, the economy is asymptotically bubbleless: it converges to the bubbleless steady state. For $b_{t_0} = \bar{b} \left(\int \rho_0^k i_{t_0-1}^k dk \right)$, the economy is asymptotically bubbly: it converges to the bubbly steady state.
(ii) for all $t \geq t_0$, $\frac{di_t^k}{i_t^k db_{t_0}}$ is decreasing in k .

3.3 Tests for bubbles and dynamic efficiency

Dynamic efficiency: Abel et al's test versus interest rate tests. Abel et al's (1989) test of dynamic efficiency involves comparing the value of resources used for investment every period to the value of resources produced. It is believed to be superior to an interest rate test involving a comparison of the interest rate \tilde{r} and the growth rate n since it is hard in practice to determine which interest rate to use in this comparison. We will see that in our model, $\tilde{r} < n$ does not necessarily imply that the economy is dynamically inefficient.

Consider a steady state, with or without bubbles, where investment and interest rates are given by \tilde{i} and \tilde{r} . In steady state, resources being used for investment in period t , or equivalently the total wealth of generation t at birth, normalized by the population of generation $t-1$ are $A(1+n) + (1-\theta)l(1+n) / (1+\tilde{r})$. In steady state, resources being produced from investment in period t , normalized by the population of generation t are $\rho_1 \tilde{i} / (1+n) + l$. Hence Abel et al's criterion tests whether or not

$$\frac{\rho_1 \tilde{i}}{1+n} + l - A(1+n) - (1-\theta)l \frac{1+n}{1+\tilde{r}} > 0$$

or equivalently, in a bubbleless steady state,

$$\frac{(\rho_1 - \rho_0) i^*}{1+n} + \left(A + \frac{(1-\theta)l}{1+r^*} \right) (r^* - n) > 0 \quad (\text{DE})$$

Hence we can have $r^* < n$ and still Abel et al's test accepting dynamic efficiency (DE). Note, further, that if all output were pledgeable and therefore the rate of return were equal for internal and external funds ($\rho_0 = \rho_1$), then (DE) would boil down to the standard comparison between the rate of interest and the rate of growth.

Bubbles and dynamic efficiency. We are interested in the predictive content of dy-

dynamic efficiency tests for the presence or the possibility of bubbles. In the standard model of Tirole (1985), bubbles can arise only if the bubbleless steady state is dynamically inefficient. In that case the bubbly steady state is dynamically efficient and all asymptotically non bubbly paths are dynamically inefficient. Therefore, if the actual economy is found to be dynamically inefficient, bubbles are possible. If on the other hand the economy is dynamically efficient, then either bubbles are impossible or we are on an asymptotically bubbly path. Moreover, dynamic efficiency can be assessed by judging whether or not $r^* > n$.

In our model, the link between bubbles and dynamic efficiency is considerably weakened. The possibility of bubbles – i.e. (B) – is consistent with the bubbleless steady state being either dynamically efficient or inefficient. The possibility of bubbles is still determined however, by the interest rate test $r^* < n$. In addition, our model sheds some light as to which interest rate to use in this test: the rate that should be used corresponds to an "uninformed" interest rate – a relatively low interest rate. Thus the considerations brought about by our analysis go part of the way towards rehabilitating interest rate tests as an indication for the possibility of bubbles.

Bubbleless steady state. Let us first consider the bubbleless steady state. We know that (B) is equivalent to $r^* < n$. Hence if (B) doesn't hold, then the bubbleless steady state is dynamically efficient. Dynamic efficiency of the bubbleless steady state, however, is consistent with (B) and therefore does not preclude the existence of a bubbly steady state.⁷

⁷This can be seen most clearly in the case where $l = 0$. We can rewrite (DE) as

$$\frac{(\rho_1 - \rho_0) A(1+n)}{1+n} \frac{1}{1 - \frac{\rho_0}{1+n}} + A \left(\frac{\rho_0}{1 - \frac{\rho_0}{1+n}} - (1+n) \right) > 0$$

which reduces to

$$\rho_1 + \rho_0 > 1 + n \tag{7}$$

Similarly, one can see that (B) reduces to

$$2\rho_0 < 1 + n$$

We can rewrite the condition for dynamic efficiency as

$$[A(1+r^*) + (1-\theta)l] \frac{\frac{r^*-n}{1+r^*} + \frac{\rho_1-\rho_0}{\rho_0}}{\frac{\rho_1-\rho_0}{\rho_0}} > l$$

It appears that if $\frac{r^*-n}{1+r^*} + \frac{\rho_1-\rho_0}{\rho_0} < 0$, then the economy is clearly dynamically inefficient. Suppose that $r^* < n$ (otherwise the bubbleless steady state is dynamically efficient): for a given $r^* < n$, the condition for dynamic efficiency is more likely to be verified if ρ_0 is smaller. However a smaller ρ_0 also implies a lower r^* which makes the condition for dynamic efficiency less likely to be verified. The overall effect of ρ_0 is therefore ambiguous when $r^* < n$. When ρ_0 is high enough however, then $r^* \geq n$, and the economy is dynamically efficient.

Bubbly steady state. Let us now consider the bubbly steady state. The bubbly steady state is dynamically efficient if and only if

$$\Lambda \geq \frac{1 - \frac{\rho_1 + \rho_0}{1+n}}{1 - \frac{\rho_0}{1+n}}$$

Note that the right hand side is always negative since we have assumed that investors prefer to invest rather than roll over their liquidity: $\rho_1 \geq 1+n$. Therefore as in Tirole (1985), the bubbly steady state is dynamically efficient. In that sense, bubbles improve the efficiency of the economy.

Pareto improvements. Bubbles need not generate Pareto improvements when the economy is inefficient. It turns out that when $l = 0$ a bubble given to the generation about to invest (generation t at date $t+1$) has absolutely no effect on investment. The interest rate increases up to the point where the equity multiplier has decreased enough to leave investment unaffected. A bubble given to consumers would ameliorate the short run and deteriorate the long run. More generally, we saw that the bubble always (weakly) increased investment. It cannot, therefore, generate a Pareto improvement: in the first period, less resources will be left for consumption.

Proposition 5 *The higher rate of return on internal funds than on borrowed ones implies that dynamic efficiency (in the sense of Abel et al) is consistent with bubbleless rates of interest below the rate of growth of the economy, and with the existence of asymptotically*

bubbly paths. The possibility of bubbles is exactly determined by an (uninformed investor) interest rate test of the form $r^ < n$.*

3.4 Consumers with positive net demand for stores of values

So far, we have assumed that consumers of generation t consume only at date t and are therefore net suppliers of stores of values. We now assume that consumers have a positive net demand for stores of value: $\theta < 0$.

Note that in this case, there are two intersections of the $i_t = i_{t-1}$ locus with $b_t = 0$:

$$0 = i_{t-1}^2 \left(1 - \frac{\rho_0}{1+n} \right) - i_{t-1} [(1+n)A + (1-\theta)l + [2 - \frac{(1+n)}{\rho_0}] \theta l] - \frac{\theta l^2 (1+n)}{\rho_0}.$$

The highest solution corresponds to the unique bubbleless steady state. The lowest solution does not correspond to a valid steady state since it is associated with a negative yield $1+r < 0$.⁸ Note that the bubbleless steady state is stable just as in the case $\theta > 0$.

Note also that when $\theta < 0$, there are two intersections between the b_t^i and b_t^b schedules with investment levels given respectively by

$$\frac{1 - \frac{\rho_0}{1+n}}{\frac{\rho_0}{(1+n)}} (-\theta l) \quad \text{and} \quad \frac{(1-\theta)l + A(1+n)}{1 - \frac{\rho_0}{1+n}}$$

Throughout the section, we maintain the assumption that

$$-\theta \Lambda < \frac{\frac{\rho_0}{1+n}}{(1 - \frac{\rho_0}{1+n})^2} \tag{8}$$

so that the bubbly steady state corresponds to the higher of the two solutions to the equation in i : $b_t^i(i) = b_t^b(i)$. Condition (8) is more likely to be verified, the lower the level of rents l , the higher the level of pledgeable income ρ_0 and the higher the net worth

⁸When $\theta < 0$, the system of equations defining bubbleless steady states has two solutions, only one of them carrying a positive yield $1+r^*$:

$$\begin{aligned} i^* &= \frac{l}{1 - \frac{\rho_0}{1+r^*} - \frac{\rho_0}{1+n}} \\ i^* &= \frac{1+n}{\rho_0} [A(1+r^*) - \theta l] \end{aligned}$$

of entrepreneurs A .

When this condition is violated the bubbly steady state becomes unstable. The crucial difference is that in this case, a perturbation of the bubbly steady state with $db_0 = 0$ and $di_{-1} < 0$ leads to $di_0 < di_{-1}$. Similarly, a perturbation with $db_0 = 0$ and $di_{-1} > 0$ leads to $di_0 > di_{-1}$. Intuitively, there is so much net worth and so little collateral in the economy that investment dynamics become unstable. There is a snowball effect on investment: more investment increases collateral which in turn increases investment even more.

Proposition 6 *Assume that (B) holds, that $\theta < 0$, and that (8) holds. Then $r^* < n$ and $i^{**} < i^*$. There exists $\underline{i} < i^{**}$ such that for any $i_{t_0-1} \geq \underline{i}$, there exists a maximum feasible bubble $\bar{b}(i_{t_0-1})$. Investment decreases with the size of the bubble. For $b_{t_0} < \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbleless: it converges to the bubbleless steady state. For $b_{t_0} = \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbly: it converges to the bubbly steady state.*

Bubbles increase the interest rate and induce a transfer from borrowers to lenders. When $\theta > 0$, the non-corporate sector is a net borrower. The bubble then operates a transfer from the non-corporate sector to the corporate sector, which increases investment. *Bubbles and investment are complements.* When $\theta < 0$, the opposite happens and bubbles crowd investment out. *Bubbles and investment are then substitutes.*

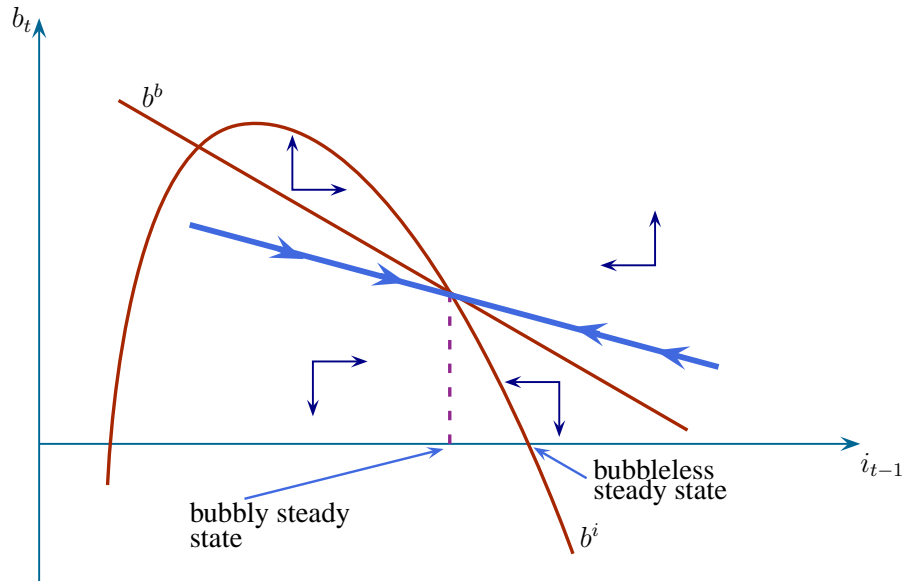


Figure 3: Phase diagram when consumer sector is net lender

3.5 Governance

A key theme in corporate finance is that firms can boost pledgeable income (ρ_0) at the cost of a sacrifice in value (ρ_1).⁹ For example, they can pledge more collateral, creating moral hazard, monitoring costs and reduced flexibility; they can enlist a private monitor (venture capitalist, large shareholder, bank); or they can abandon private equity for a public listing, at the cost of transparency obligations, reduced incentives and so forth. The trade-off between pledgeable income and value can be formalized by a decreasing function ($\rho_1 = H(\rho_0)$).

Let ρ_1^t and ρ_0^t denote the values chosen by generation t , the utility of generation t entrepreneurs is

$$\begin{aligned} U_t &= (\rho_1^t - \rho_0^t)i_t \\ &= \left[\frac{\rho_1^t - \rho_0^t}{1 - \frac{\rho_0^t}{1+r_{t+1}}} \right] \left[b_t + l + \frac{\rho_0^{t-1} i_{t-1}}{1+n} \right] \end{aligned}$$

Thus, when liquidity is scarce (the interest rate is low), firms will sacrifice value in order to boost pledgeable income.

To avoid re-analyzing the complete path, let us assume that governance choices are made “at the margin” and so the paths described in Propositions 1 and 2 are approximations of the realized paths with endogenous governance. The optimal governance choice is determined by maximizing, in every period t ,

$$\frac{H(\rho_0^t) - \rho_0^t}{1 - \frac{\rho_0^t}{1+r_{t+1}}}.$$

Proposition 7 *A bubble, by increasing interest rates, reduces the benefits of creating pledgeable income. It is therefore conducive to looser governance (lower ρ_0^t , higher ρ_1^t).*

4 Stochastic bubbles

4.1 Bubbly liquidity discount

As in Weil (1987), we can allow the bubble to burst stochastically. Suppose that each period the bubble bursts with probability $1 - \lambda$.

⁹See, e.g., Tirole (2006) for an overview.

An asset's liquidity service depends on what the asset delivers when cash is particularly valuable to firms. Building on this idea,¹⁰ we now argue that, even in this risk neutral, constant-returns-to-scale (CRS) environment, a stochastic bubble trades at a liquidity discount (or equivalently in this model an equity premium) relative to rents.

Let $r^*(i_t)$ denote the interest rate prevailing on the bubbleless path for "initial" investment i_t ; this function was derived in Section 2.2: $r^*(i_t) = A^{-1} \left(\frac{\rho_0}{1+n} i_t + \theta l \right) - 1$. Let i_t and r_{t+1} (respectively, i_t^- and $r^*(i_t^-)$) denote the investment levels and interest rates when the bubble has lasted until period $t+1$ and continues (respectively, bursts). These are given by:

$$i_t = \frac{b_t + l + \frac{\rho_0 i_{t-1}}{1+n}}{1 - \frac{\rho_0}{1+r_{t+1}}} \quad \text{and} \quad i_t^- = \frac{l + \frac{\rho_0 i_{t-1}}{1+n}}{1 - \frac{\rho_0}{1+r^*(i_t^-)}}$$

where the asset demand equation if the bubble has lasted until period $t+1$ and continues is unchanged

$$i_t = \frac{1+n}{\rho_0} [A(1+r_{t+1}) - \theta l - \frac{1+r_{t+1}}{1+n} b_t]$$

Since i_t^- and $r^*(i_t^-)$ are determined by the same set of equations as i_t and r_{t+1} but with $b_t = 0$, it is clear that $i_t > i_t^-$ and $r_{t+1} > r^*(i_t^-)$: the burst of the bubble depresses investment and the interest rate.

At date t , generation- t entrepreneurs can hold safe assets (rents, claims on previous investments' income) or risky ones (stochastic bubble). Letting \tilde{r}_t denote the return on the bubble when it does not burst, the arbitrage equation is:

$$\lambda \frac{1+r_t}{1 - \frac{\rho_0}{1+r_{t+1}}} + (1-\lambda) \frac{1+r_t}{1 - \frac{\rho_0}{1+r^*(i_t^-)}} = \lambda \frac{1+\tilde{r}_t}{1 - \frac{\rho_0}{1+r_{t+1}}} \quad (9)$$

This in turn implies that $1+\tilde{r}_t > (1+r_t)/\lambda$. Despite risk neutrality and CRS, stochastic bubbles trade at a positive discount $(1+r_t)^{-1} - \lambda^{-1} (1+\tilde{r}_t)^{-1}$ – alternatively, they command positive net expected returns $\lambda(1+\tilde{r}_t) - (1+r_t)$. The intuition is straightforward. Bubbles deliver no income when liquidity is scarce and so the interest rate is low, implying that internal funds can be levered substantially.

A steady state along the bubbly path is given by $\tilde{r}^{**} = n$ and

¹⁰Which is but the transposition on the production side of standard (CAPM) principles on the consumption side (see e.g., Holmström-Tirole 2001).

$$1 + r^{**} = \frac{1+n}{1 + \frac{1-\lambda}{\lambda} \frac{1 - \frac{\rho_0}{1+r^{**}}}{1 - \frac{\rho_0}{1+r^{**}(i^{**})}}}$$

$$i^{**} = \frac{\lambda(1+n) + 1 - \theta \frac{1+n}{1+r^{**}}}{1 - \frac{\rho_0}{1+n}}$$

$$b^{**} = i^{**} \left(1 - \frac{\rho_0}{1+r^{**}} - \frac{\rho_0}{1+n} \right) - 1$$

The condition for the existence of a bubble becomes

$$i^{**} \left(1 - \frac{\rho_0}{1+r^{**}} - \frac{\rho_0}{1+n} \right) - 1 > 0 \quad (B'')$$

Unfortunately, this condition determines only implicitly the parameter region that leads to the possibility of bubbles. It features two endogenous objects, i^{**} and r^{**} that are the solutions to the non-linear system of equations above. This complication arises for the following reason. Bubbles now present a risk premium and thus a positive net expected return: $\lambda(1 + \tilde{r}^{**}) - (1 + r^{**}) > 0$. In a bubbly steady state, zero per capita bubble growth pins down the expected return on bubbles: $\lambda(1 + \tilde{r}^{**}) = n$; both the risk-free rate r^{**} and investment i^{**} have to be determined jointly as solutions to a non-linear system.

Steady state investment i^{**} , bubble size b^{**} and interest rate r^{**} are all decreasing in the probability that the bubble crashes $1 - \lambda$. A more stable bubble provides more liquidity and is more conducive to investment. This in turn boosts the demand for liquidity and makes for a larger bubble.

Proposition 8 *Suppose that the consumer sector is a net borrower ($\theta > 0$) and that (B'') holds, then: (i) bubbles trade at a liquidity discount; (ii) in steady state along the bubbly path – before the bubble bursts – as, the probability of bursting ($1 - \lambda$) increases, investment i^{**} and bubble b^{**} decrease; (iii) as long as the bubble lasts, investment is high and interest rates are high; (iv) when the bubble bursts, investment immediately decreases and keeps decreasing until we reach the bubbleless steady state; (v) the bursting of the bubble makes collateral more valuable: firms scramble for collateral.*

The dynamics when the bubble still lasts are more complicated. Indeed, an extra state variable is now required to describe the economy. The state space is now given by

the triple $(\dot{i}_{t-1}, \mathbf{b}_t, r_t)$. The reason past interest rates r_t have to be kept track of is that the arbitrage equation (9) involves both the interest rate at date t and at date $t + 1$. As a consequence, phase diagrams cannot be used anymore. A full characterization of the stability properties of the different steady states and their basin of attraction is rather involved and outside the scope of this paper.

4.2 Bubbles bursting endogenously

We now modify the environment in the following way. Suppose that A follows a two-state Markov process $A \in \{A_H, A_L\}$ with $A_H > A_L$. Initially $A = A_H$. With probability $1 - \lambda > 0$ per period, A transitions to A_L which is an absorbing state.

We assume that (B'') is verified for an exogenous bursting probability of $(1 - \lambda)$ in an economy with a deterministic and constant A equal to A_H . Similarly, we assume that for any $\lambda \in [0, 1]$, (B'') is violated for an exogenous bursting probability of $(1 - \lambda)$ in an economy with a deterministic and constant A equal to A_L . Hence if the economy is already in the state of low net worth $A = A_L$, the demand for liquidity is low, the interest rate is high and bubbles cannot exist. Bubbles however, can exist as long as net worth is high: $A = A_H$. For notation simplicity, we keep the notation $r^*(\dot{i}_t)$ for the interest rate prevailing on the bubbleless path for ‘‘initial’’ investment \dot{i}_t when $A = A_L$.¹¹

Suppose that $A_t = A_H$ and consider the economy entering period $t + 1$ with a bubble of size \mathbf{b}_t and a capital stock given by \dot{i}_{t-1} . Then if $A_{t+1} = A_H$, $(\mathbf{b}_{t+1}, \dot{i}_t)$ are given by the same equations as in Section 4.2.1 with $A = A_H$. On the other hand if $A_{t+1} = A_L$, then the bubble bursts: $\mathbf{b}_t = 0$ and \dot{i}_t is given by the same equations as above with $A = A_L$ and $\mathbf{b}_t = 0$. The economy then evolves as in Section 2.2.

Proposition 9 *Assume that $\theta > 0$ and that (B'') holds with $A = A_H$. Consider an economy where in the initial period the economy is in steady state along the bubbly path and $A_t = A_H$. In the first period where $A_t = A_L$, the bubble bursts, investment drops, and the economy converges to the bubbleless steady state corresponding to $A = A_L$.*

¹¹Formally, $r^*(\dot{i}_t)$ is determined by

$$1 + r^*(\dot{i}_t) = A_L^{-1} \left(\frac{\rho_0}{1 + n} \dot{i}_t + \theta l \right).$$

This environment makes clear that bad shocks to corporate balance sheets can potentially have an amplified effect on investment over and above that described in the literature emphasizing the importance of corporate net worth – for example Kiyotaki-Moore (1997) – by triggering liquidity dry-ups in the form of bubble bursts.

5 Robustness

5.1 Concave preferences and unconstrained firms

In this section, we adopt the setup where consumers have concave preferences as in Section 2.4.1 and generalize our comparative statics results to that case. Note that the model with unconstrained firms described in Section 2.4.1 is nested by the model where consumers have concave preferences. The analysis below therefore also applies to the setup with unconstrained firms (see the remark below).

Let L be the endowment of rents per entrepreneur. Both the level of rents and the fraction of rents in the hands of consumers now depends on the interest rate. The supply and demand equations for stores of values are now

$$\dot{i}_t = \frac{\rho_0 \dot{i}_t}{1 + r_{t+1}} + l_t + L + b_t + \frac{\rho_0 \dot{i}_{t-1}}{1 + n} \text{ and } \dot{i}_t = \frac{1 + n}{\rho_0} [A(1 + r_{t+1}) - l_{t+1} - \frac{1 + r_{t+1}}{1 + n} b_t]$$

The steady analysis, however, remains very tractable. In a bubbly steady state

$$\begin{aligned} i^{**} &= \frac{L + A(1 + n)}{1 - \frac{\rho_0}{1+n}} \\ b^{**} &= [L + A(1 + n)] \frac{1 - 2\frac{\rho_0}{1+n}}{1 - \frac{\rho_0}{1+n}} - l(n) - L \\ r^{**} &= n \end{aligned}$$

Note that investment in a steady state with bubbles is independent of the function $l(r_t)$.¹² There is perfect crowding out between bubbles and rents created by consumers: $b^{**} + l(n)$ is independent of the function $l(r_t)$.

By contrast, as we already noted, rents in the hands of entrepreneurs L increase

¹²In particular, some of the microfoundations we provided for rents led to a similar decreasing function $l(r_t)$. The analysis that is conducted in this section applies to these cases as well.

entrepreneurs' net worth and hence investment. As a result, they do not completely crowd out bubbles: $\mathbf{b}^{**} + L$ is increasing in L .

The condition for the bubbly steady state becomes

$$[L + A(1 + n)] \frac{1 - 2\frac{\rho_0}{1+n}}{1 - \frac{\rho_0}{1+n}} - l(n) > 0 \quad (B')$$

Proposition 10 *Suppose that (B') holds. Then $r^* < n$.*

(i) *If $l(r^*) > 0$, investment in the bubbly steady state i^{**} is higher than investment i^* in the non-bubbly steady state.*

(ii) *If $l(r^*) < 0$ and $-\frac{1 - \frac{\rho_0}{1+n}}{\frac{\rho_0}{1+n}} l(r^*) < \frac{L + A(1+n)}{1 - \frac{\rho_0}{1+n}}$, investment in the bubbly steady state i^{**} is lower than investment i^* in the non-bubbly steady state.*

Therefore, we verify once again that bubbles crowd investment in as long as consumers are net suppliers of rents. The assumption that $l(r^*) > 0$ is crucial to determine whether bubbles are complements or substitutes. Bubbles raise the interest rate and transfer wealth from borrowers to lenders. When $l(r^*) > 0$, consumers are borrowers. The bubble then operates a transfer from consumers to entrepreneurs, which increases investment. Bubbles and rents are complements. When $l(r^*) < 0$, the opposite happens and bubbles crowd investment out. Bubbles and rents are then substitutes.

Remark: Note that this analysis nests the model with unconstrained firms described in Section 2.4.1. In that model, we have $l(r^*) = f(k^*) > 0$. Proposition 10 then shows that $i^{**} > i^*$. However, note that the steady state investment level for unconstrained firms in a bubbly steady-state is lower than in the non-bubbly steady state: $k^{**} < k^*$. This is the standard crowding out effect of bubbles on investment. Therefore, bubbles crowd the investment of constrained firms in $i^{**} > i^*$ but crowds the investment of unconstrained firms out $k^{**} < k^*$.

5.2 Consumers with linear preferences

We now examine the case where consumers are able to substitute present consumption for future consumption with a per-period discount factor β . There are two cases to consider. If $\beta(1 + r_{t+1}) < 1$, then the dynamics are characterized by the same equations we have

been using so far: equations (5) and (6). If $\beta(1 + r_{t+1}) \geq 1$, on the other hand, then the analysis has to be modified as consumers will hold on to at least part of their endowment of rents. We will assume that $\beta(1 + n) < 1$, so that the bubbly steady state characterized above exists. Appendix A3 characterizes the dynamics of the economy and the phase diagram, depicted in Figure 4.

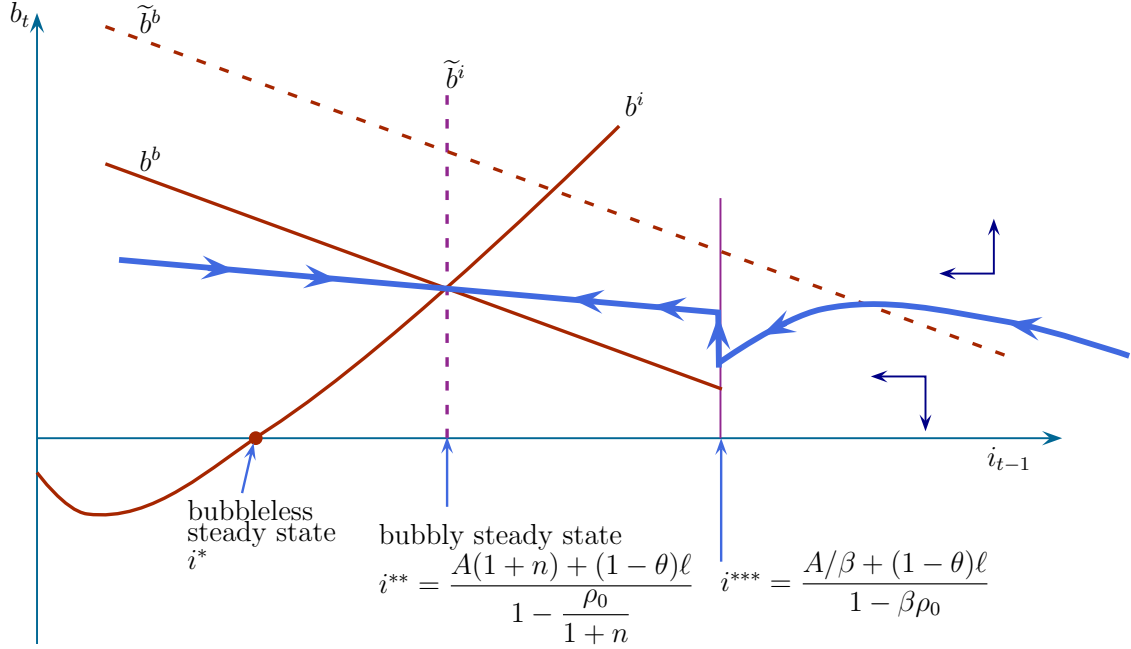


Figure 4: Phase diagram with linear consumer preferences

Proposition 11 *Suppose that consumers have linear preferences with discount factor β , that (B) holds and that $\beta(1 + n) < 1$. Suppose further that liquidity initially is plentiful due to previous investment (i_{t-1} high). The economy evolves through three consecutive phases. First, consumers do not sell their rents, which yield high interest rates; investment decreases; the bubble increases and then decreases. Second, investment remains constant, consumers sell some of their rents and the bubble increases. Finally, consumers sell all their rents and the bubble increases and investment decreases to the steady state, as described in Proposition 2.*

5.3 Producers with linear preferences

Finally, we assumed for convenience that entrepreneurs consume only when old. Suppose by contrast that they have linear preferences, with discount factor β . Focus on the region

in which

$$i_{t-1} < \frac{A/\beta + (1 - \theta)l}{1 - \beta\rho_0}$$

for which we know from the previous analysis that consumers would not want to compete with firms for stores of value; neither would the entrepreneurs want to save unless those savings are used for future investment.

We need to check that entrepreneurs prefer investing in stores of value and then in productive investment to consuming when young:

$$\beta^2(\rho_1 - \rho_0)i_t > A$$

or, using

$$i_t = \frac{A(1 + r_t)}{1 - \frac{\rho_0}{1 - r_{t+1}}}$$

$$\beta^2(1 + r_t)\rho_1 > 1 - \rho_0\left[\frac{1}{1 + r_{t+1}} - \beta^2(1 + r_t)\right] \quad (10)$$

Condition (10), which ensures that our analysis extends to producers with linear preferences, is easier to satisfy when ρ_1 and (given that $\beta(1 + r_t)$ and $\beta(1 + r_{t+1})$ are smaller than 1) ρ_0 increase.

6 Conclusion

This paper has made several contributions.

First, we have studied the interplay between inside liquidity and outside liquidity (stores of value). Outside liquidity helps firms address the asynchronicity between their access to and need for cash – the liquidity effect– but also compete for savings with productive investment – the competition effect. The liquidity effect dominates when inside liquidity is abundant, and when outside liquidity is outside of an intermediate region.

Second, we have shown that bubbles are more likely to exist and can be larger when inside and outside liquidity are scarce. Dynamic efficiency is consistent with the existence of bubbles, provided that the rate of return on internal funds exceeds that on borrowed ones, i.e. provided that capital markets are imperfect.

Third, bubbles are a form of outside liquidity. They crowd the financially constrained corporate sector's investment in (out) if the unconstrained sector is a net borrower (lender). The burst of a bubble has a negative financial wealth effect on firms, and further reduces liquidity. The former effect is contractionary, and so is the latter if the unconstrained sector is a net borrower. Conversely, permanent real wealth losses by firms make it harder to sustain a bubble, and so financial disturbances amplify real ones. Even in a risk neutral, constant-returns-to-scale environment, a stochastic bubble trades at a liquidity discount relative to rents since it pays more in states where the equity multiplier is low. Finally, bubbles impact other corporate decisions as well. In particular, firms are predicted to strengthen their governance in periods of scarce liquidity, as when a bubble has burst.

Our analysis brings support to the idea that pricking bubbles may be hazardous. But it also suggests when this will be particularly so, namely when the unconstrained sector is a net borrower. To go further, though, one will need to not only analyze the impact of the popping up of the bubble, but also the way it is performed, for example a sustained increase in the interest rate or an increase in the supply of public liquidity, which provided that it is not matched by an equivalent reduction in consumer's supply of liquidity will hamper the continuation of the bubble. Extending the microfoundations of Section 2.4.1 and adding a public supply of liquidity should therefore stand high on our research agenda.

Our imperfect capital-markets analysis implies a divergence between the rates of returns on internal and borrowed funds, and therefore that rates of return on borrowed funds below the rate of growth of the economy are consistent with Abel et al (1989)'s finding that the productive sector may disgorge at least as much as it invests. The out-flow measure in Abel et al aggregates a variety of firms with wildly different governance structures and therefore pledgeable income (publicly traded firms, family and private equity, startups). Using the theoretical analysis to build a modified version of Abel et al's clever test of potential existence of bubbles would be of much interest as well.

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Appendix

A.1 Proofs for Section 2

Derivation of investment i^ in the bubbleless steady state.* We have

$$i^* = \frac{A(1+n) + (1-\theta)l - \theta l \left(\frac{1+n}{\rho_0} - 2 \right) + \sqrt{\left[A(1+n) + (1-\theta)l - \theta l \left(\frac{1+n}{\rho_0} - 2 \right) \right]^2 + 4\theta l^2 \left(\frac{1+n}{\rho_0} - 1 \right)}}{2 \left(1 - \frac{\rho_0}{1+n} \right)}$$

Similarly one can compute the steady state interest rate r^* :

$$1 + r^* = \frac{\theta l + \frac{\rho_0}{1+n} i^*}{A}$$

$$x = \frac{1+n}{\rho_0}.$$

Proof of Lemma 1. We can solve for investment i_t as a function of i_{t-1} and b_t :

$$i_t = \frac{(1+n)A + \frac{\rho_0 i_{t-1}}{1+n} + \left[1 - \frac{\theta(1+n)}{\rho_0} \right] l + \sqrt{\left\{ (1+n)A + \frac{\rho_0 i_{t-1}}{1+n} + \left[1 - \frac{\theta(1+n)}{\rho_0} \right] l \right\}^2 + 4 \frac{\theta(1+n)l}{\rho_0} \left[b_t + l + \frac{\rho_0 i_{t-1}}{1+n} \right]}}{2}$$

From this expression, it is clear that i_t is increasing in i_{t-1} and b_t .

Proof of Proposition 1. The steady state is given by

$$\begin{aligned} i^* &= \frac{l}{1 - \frac{\rho_0}{1+r^*} - \frac{\rho_0}{1+n}} \\ i^* &= \frac{1+n}{\rho_0} [A(1+r^*) - \theta l] \end{aligned}$$

which can be solved for as

$$i^* = \frac{A(1+n) + (1-\theta)l - \theta l \left(\frac{1+n}{\rho_0} - 2 \right) + \sqrt{\left[A(1+n) + (1-\theta)l - \theta l \left(\frac{1+n}{\rho_0} - 2 \right) \right]^2 + 4\theta l^2 \left(\frac{1+n}{\rho_0} - 1 \right)}}{2 \left(1 - \frac{\rho_0}{1+n} \right)}$$

Similarly one can compute the steady state interest rate r^* :

$$1 + r^* = \frac{\theta l + \frac{\rho_0}{1+n} i^*}{A}$$

We can therefore express $2 \left(1 - \frac{\rho_0}{1+n}\right) \frac{\partial i^*}{\partial l}$ as

$$\begin{aligned} &= 1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) + \frac{2 \left[A(1+n) + l - \theta l \left(\frac{1+n}{\rho_0} - 1 \right) \right] \left[1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) \right] + 8\theta l \left(\frac{1+n}{\rho_0} - 1 \right)}{2\sqrt{\left[A(1+n) + l - \theta l \left(\frac{1+n}{\rho_0} - 1 \right) \right]^2 + 4\theta l^2 \left(\frac{1+n}{\rho_0} - 1 \right)}} \\ &= \frac{2 \left[\sqrt{\left[A(1+n) \right]^2 + 2A(1+n)l \left[1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) \right] + l^2 \left[1 + \theta \left(\frac{1+n}{\rho_0} - 1 \right) \right]^2} \right.}{2\sqrt{\left[A(1+n) + l - \theta l \left(\frac{1+n}{\rho_0} - 1 \right) \right]^2 + 4\theta l^2 \left(\frac{1+n}{\rho_0} - 1 \right)}} \left. \left[1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) \right] \right. \\ &\quad \left. + \frac{8\theta l \left(\frac{1+n}{\rho_0} - 1 \right)}{2\sqrt{\left[A(1+n) + l - \theta l \left(\frac{1+n}{\rho_0} - 1 \right) \right]^2 + 4\theta l^2 \left(\frac{1+n}{\rho_0} - 1 \right)}} \right] \end{aligned}$$

Suppose that $1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) \geq 0$, then clearly $\frac{\partial i^*}{\partial l} > 0$ for all $l \geq 0$. Suppose now that $1 - \theta \left(\frac{1+n}{\rho_0} - 1 \right) \leq 0$. The condition that $\frac{\partial i^*}{\partial l} \geq 0$ can be simplified to

$$\begin{aligned} l &\geq \frac{2A(1+n) \frac{\rho_0}{1+n} \left(1 - \frac{\rho_0}{1+n} \frac{1+\theta}{\theta} \right)}{4 \frac{\rho_0}{1+n} \left(1 - \frac{\rho_0}{1+n} \right) + \theta \left(1 - \frac{\rho_0}{1+n} \frac{1+\theta}{\theta} \right)^2} \\ l &\geq \frac{2A(1+n) \frac{\rho_0}{1+n} \left(1 - \frac{\rho_0}{1+n} \frac{1+\theta}{\theta} \right)}{\theta \left(1 + \frac{\rho_0}{1+n} \frac{1-\theta}{\theta} \right)^2} \end{aligned}$$

or

$$l \geq l_0 \left(\frac{\rho_0}{1+n}, A, 1+n, \theta \right)$$

where

$$l_0 \left(\frac{\rho_0}{1+n}, A, 1+n, \theta \right) \equiv \frac{2A(1+n) \frac{\rho_0}{1+n} \left(1 - \frac{\rho_0}{1+n} \frac{1+\theta}{\theta} \right)}{\theta \left(1 + \frac{\rho_0}{1+n} \frac{1-\theta}{\theta} \right)^2}$$

■

A.2 Proofs for Section 3

Dynamics in section 3.1. The $i_t = i_{t-1}$ schedule is given by

$$b_t = i_{t-1}^2 \frac{\rho_0}{\theta l(1+n)} \left(1 - \frac{\rho_0}{1+n}\right) - \frac{\rho_0 i_{t-1}}{\theta l(1+n)} [(1+n)A + (1-\theta)l + [2 - \frac{(1+n)}{\rho_0}]\theta l] - l.$$

This defines the schedule as a function b_t of i_{t-1} : $b_t^i(i_{t-1})$.

The $b_t = b_{t+1}$ schedule is given by

$$\begin{aligned} & A(1+n) \left(2\frac{1+n}{\rho_0} - 1\right) - \frac{\rho_0 i_{t-1}}{1+n} - [1 + \frac{\theta(1+n)}{\rho_0}]l - \frac{2(1+n)}{\rho_0} b_t \\ = & \sqrt{\left\{ (1+n)A + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l \right\}^2 + 4\frac{\theta(1+n)l}{\rho_0} [b_t + l + \frac{\rho_0 i_{t-1}}{1+n}]} \\ & \frac{2(1+n)}{\rho_0} [A(1+n) - \theta l - b_t] - [A(1+n) + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l] \\ = & \sqrt{\left\{ (1+n)A + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l \right\}^2 + 4\frac{\theta(1+n)l}{\rho_0} [b_t + l + \frac{\rho_0 i_{t-1}}{1+n}]} \\ & \left\{ \frac{2(1+n)}{\rho_0} [A(1+n) - \theta l - b_t] \right\}^2 + [A(1+n) + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l]^2 \dots \\ & - 4\frac{(1+n)}{\rho_0} [A(1+n) - \theta l - b_t] [A(1+n) + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l] \\ = & \left\{ (1+n)A + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l \right\}^2 + 4\frac{\theta(1+n)l}{\rho_0} [b_t + l + \frac{\rho_0 i_{t-1}}{1+n}] \\ & \frac{(1+n)}{\rho_0} [A(1+n) - \theta l - b_t]^2 - [A(1+n) - \theta l - b_t] [A(1+n) + \frac{\rho_0 i_{t-1}}{1+n} + [1 - \frac{\theta(1+n)}{\rho_0}]l] \\ = & \theta l [b_t + l + \frac{\rho_0 i_{t-1}}{1+n}] \end{aligned}$$

$$\frac{\frac{(1+n)}{\rho_0} [A(1+n) - \theta l - b_t]^2 - \theta l [b_t + l] - [A(1+n) - \theta l - b_t] \left[A(1+n) + \left[1 - \frac{\theta(1+n)}{\rho_0} \right] l \right]}{\theta l \frac{\rho_0}{1+n} [A(1+n) - b_t]} = i_{t-1}$$

$$X_t = -A(1+n) + \theta l + b_t$$

$$\frac{\frac{1+n}{\rho_0} X_t^2 + \left[A(1+n) + (1-\theta)l - \frac{1+n}{\rho_0} \theta l \right] X_t - \theta l [(1-\theta)l + A(1+n)]}{\frac{\rho_0}{1+n} [\theta l - X_t]} = i_{t-1}$$

$$\frac{\frac{1+n}{\rho_0} (X_t - \theta l) \left[X_t + \frac{\rho_0}{1+n} [(1-\theta)l + A(1+n)] \right]}{\frac{\rho_0}{1+n} (\theta l - X_t)} = i_{t-1}$$

$$\left(\frac{1+n}{\rho_0} \right)^2 \left[-X_t - \frac{\rho_0}{1+n} [(1-\theta)l + A(1+n)] \right] = i_{t-1}$$

$$\left(\frac{1+n}{\rho_0} \right)^2 \left[\left(1 - \frac{\rho_0}{1+n} \right) [A(1+n) + (1-\theta)l] - b_t - l \right] = i_{t-1}$$

To sum up

$$b_t^i = i_{t-1}^2 \frac{\rho_0}{\theta l (1+n)} \left(1 - \frac{\rho_0}{1+n} \right) - \frac{\rho_0 i_{t-1}}{\theta l (1+n)} [(1+n)A + (1-\theta)l + \left[2 - \frac{(1+n)}{\rho_0} \right] \theta l] - l.$$

$$b_t^b = - \left(\frac{\rho_0}{1+n} \right)^2 i_{t-1} + \left(1 - \frac{\rho_0}{1+n} \right) [A(1+n) + (1-\theta)l] - l$$

We have

$$b_t^i(0) = -l$$

$$b_t^b(0) = \left(1 - \frac{\rho_0}{1+n} \right) [A(1+n) + (1-\theta)l] - l > 0 \text{ from the bubble existence condition}$$

$$\frac{db_t^i}{di_{t-1}} = 2i_{t-1} \frac{\rho_0}{\theta l(1+n)} \left(1 - \frac{\rho_0}{1+n}\right) - \frac{\rho_0}{\theta l(1+n)} [(1+n)A + (1-\theta)l + [2 - \frac{(1+n)}{\rho_0}]\theta l].$$

$$\begin{aligned} \frac{db_t^i}{di_{t-1}}|_{i_{t-1}=0} &= -\frac{\rho_0}{\theta l(1+n)} [(1+n)A + (1-\theta)l + [2 - \frac{(1+n)}{\rho_0}]\theta l] \\ &= -\frac{\rho_0}{\theta l(1+n)} [(1+n)A + \left(1 + \theta - \frac{(1+n)}{\rho_0}\theta\right) l] \end{aligned}$$

which can be positive or negative.

Proof or Proposition 6. Let $L \equiv (1-\theta)l$ and define $\tilde{l} \equiv -\theta l$. The \tilde{b}^i and \tilde{b}^b schedules are now

$$\begin{aligned} b_t &= -\left(1 - \frac{\rho_0}{1+n}\right) \frac{\rho_0}{1+n} \frac{1}{\theta l} i_{t-1}^2 + \\ &\quad \frac{\rho_0 i_{t-1}}{(1+n)\theta l} [(1-\theta)l + A(1+n) + \left(1 - \frac{2\rho_0}{1+n}\right) \frac{(1+n)\theta l}{\rho_0}] - (-\theta l + (1-\theta)l) \end{aligned}$$

and

$$b_t = -\left(\frac{\rho_0}{1+n}\right)^2 i_{t-1} + \left(1 - \frac{\rho_0}{1+n}\right) A(1+n) + \theta l - \frac{\rho_0}{1+n}(1-\theta)$$

Investment is given by the following equation

$$i_t = \frac{\frac{1+n}{\rho_0} \left(1 - \frac{\rho_0}{1+n}\right) \theta l + (1-\theta)l + \frac{\rho_0 i_{t-1}}{1+n} + (1+n)A + \sqrt{\left[-\frac{1+n}{\rho_0}\theta l - \theta l + (1-\theta)l + \frac{\rho_0 i_{t-1}}{1+n} + (1+n)A\right]^2 + 4\frac{1+n}{\rho_0}\theta l [(1+n)A - b_t]}}{2}$$

Hence i_t is increasing in i_{t-1} and decreasing in b_t . Note that we have

$$\tilde{b}^b(i_{t-1}) \geq \tilde{b}^i(i_{t-1})$$

if and only if

$$\left(1 - \frac{\rho_0}{1+n}\right) \frac{\rho_0}{1+n} \frac{1}{\theta l} i_{t-1}^2 - \left[\frac{\rho_0}{1+n} \frac{1}{\theta l} ((1-\theta)l + A(1+n)) + \left(1 - \frac{\rho_0}{1+n}\right)\right] i_{t-1} + \dots$$

$$\dots + \left(1 - \frac{\rho_0}{1+n}\right) [A(1+n) + (1-\theta)l] \geq 0$$

if and only if $i_{t-1} \notin [i_2, i_1]$ where

$$\begin{aligned} i_1 &= \frac{(1-\theta)l + A(1+n)}{1 - \frac{\rho_0}{1+n}} \\ i_2 &= \frac{1 - \frac{\rho_0}{1+n}}{\frac{\rho_0}{1+n}} \theta l \end{aligned}$$

and (8) guarantees that $i_2 < i_1$. A phase diagram analysis shows that the bubbly steady state is saddle path stable, and the results in the proposition follow.

A.3 Proofs for Section 5

Proof of Proposition 10. The proof proceeds as follows. Consider the economy where rents supplied by consumers are fixed and equal to $l(r^*)$. We can use the phase diagram derived above to analyze this economy. At the intersection of b_t^i and b_t^b , investment is equal to $i^{**} = \frac{L+A(1+n)}{1-\frac{\rho_0}{1+n}}$, the bubble is equal to $\tilde{b}^{**} = [L + A(1+n)] \frac{1-2\frac{\rho_0}{1+n}}{1-\frac{\rho_0}{1+n}} - l(r^*) - L$ (possibly negative) and the interest rate is equal to n . Therefore

$$g(i^{**}, \tilde{b}^{**}) = 1+n \text{ and } g(i^*, 0) = 1+r^*$$

where $g(i, b) \equiv \frac{\frac{\rho_0 i}{1+n} + l}{A - \frac{b}{1+n}}$ is increasing in i and b . It is then easy to see that this implies $i^{**} > i^*$ if and only if $r^* < n$, and that this last condition is equivalent to $\tilde{b}^{**} > 0$. If we had $r^* > n$, then we would have $l(r^*) < l(n)$ which combined with $\tilde{b}^{**} < 0$ is in contradiction with (B'). The second part of the proposition can be proved along the same lines.

Proof of Proposition 11. Suppose then that $\beta(1+r_{t+1}) > 1$. In that case, consumers do not sell their rents. The supply and demand equations become:

$$i_t = \frac{b_t + (1-\theta)l + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \frac{\rho_0}{1+r_{t+1}}} \quad (11)$$

and

$$i_t = i^s(r_{t+1}) \equiv \frac{1+n}{\rho_0} [A(1+r_{t+1}) - \frac{1+r_{t+1}}{1+n} b_t] \quad (12)$$

A simplification occurs which allows us to solve for the dynamics in closed form:

$$\mathbf{i}_t = \mathbf{A}(1 + \mathbf{n}) + (1 - \theta)\mathbf{l} + \rho_0 \frac{\mathbf{i}_{t-1}}{1 + \mathbf{n}}$$

and

$$1 + \mathbf{r}_{t+1} = \rho_0 \frac{\mathbf{A}(1 + \mathbf{n}) + (1 - \theta)\mathbf{l} + \rho_0 \frac{\mathbf{i}_{t-1}}{1 + \mathbf{n}}}{\mathbf{A}(1 + \mathbf{n}) - \mathbf{b}_t}$$

The $\mathbf{i}_t = \mathbf{i}_{t-1}$ schedule is given by

$$\mathbf{i}_{t-1} = \frac{\mathbf{A}(1 + \mathbf{n}) + (1 - \theta)\mathbf{l}}{1 - \frac{\rho_0}{1 + \mathbf{n}}}$$

This defines a vertical schedule $\tilde{\mathbf{b}}^i$.

The $\mathbf{b}_t = \mathbf{b}_{t+1}$ schedule is given by

$$\mathbf{b}_t = - \left(\frac{\rho_0}{1 + \mathbf{n}} \right)^2 \mathbf{i}_{t-1} + [\mathbf{A}(1 + \mathbf{n}) + (1 - \theta)\mathbf{l}] \left(1 - \frac{\rho_0}{1 + \mathbf{n}} \right) - (1 - \theta)\mathbf{l}$$

which defines a schedule $\tilde{\mathbf{b}}^b(\mathbf{i}_{t-1})$.

We have $\tilde{\mathbf{b}}^b(0) = \left(1 - \frac{\rho_0}{1 + \mathbf{n}}\right) [\mathbf{A}(1 + \mathbf{n}) + (1 - \theta)\mathbf{l}] - (1 - \theta)\mathbf{l}$. It is easy to check that we always have $\tilde{\mathbf{b}}^b(\mathbf{i}_{t-1}) > \mathbf{b}_t^b(\mathbf{i}_{t-1})$. In Figure 2, $\tilde{\mathbf{b}}^b(\mathbf{i}_{t-1})$ is represented by the dotted red line. It is remarkable that the bubble \mathbf{b}_t has absolutely no impact on investment. The bubble raises the interest rate and decreases the value of rents: bubbly liquidity perfectly crowds out private liquidity.

When $\beta(1 + \mathbf{r}_{t+1}) < 1$ and consumers sell their stores of values to entrepreneurs, the apparition of the bubble raises interest rates, lowers the value of their endowment of rents, and triggers a net transfer of wealth from consumers to entrepreneurs. This in turn boosts investment. When $\beta(1 + \mathbf{r}_{t+1}) > 1$, this transfer mechanism is not operative – because consumers hold on to their rents – and there is perfect crowding out.

We still have to tackle the case where $\beta(1 + \mathbf{r}_{t+1}) = 1$. In that case, consumers are indifferent between holding on to their rents or selling them. In period $t + 1$, they will only sell a fraction \mathbf{x}_{t+1} of their rents. We then have the two equations:

$$\frac{\mathbf{A}}{\beta} = \frac{\rho_0 \mathbf{i}_t}{1 + \mathbf{n}} + \theta \mathbf{l} \mathbf{x}_{t+1} + \frac{\mathbf{b}_t}{\beta(1 + \mathbf{n})}$$

and

$$\dot{i}_t = \frac{b_t + (1 - \theta)l + \theta l x_t + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \beta \rho_0}$$

Note that in this region, the evolution of b_t is exogenous. We can solve the system using state variables (i_{t-1}, b_t, x_t) as:

$$\dot{i}_t = \frac{b_t + (1 - \theta)l + \theta l x_t + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \beta \rho_0} \quad (13)$$

$$b_{t+1} = \frac{b_t}{\beta(1+n)} \quad (14)$$

$$x_{t+1} = \frac{A(1+n) - \frac{b_t}{(1-\beta\rho_0)} - \beta\rho_0 \frac{(1-\theta)l + \theta l x_t + \rho_0 \frac{i_{t-1}}{1+n}}{1-\beta\rho_0}}{\beta(1+n)\theta l} \quad (15)$$

Several cases have to be considered depending on whether $\beta(1+r_t) > 1$ or $\beta(1+r_t) = 1$.

If $\beta(1+r_t) = 1$ for example, then $\dot{i}_t = i^{***}$ where

$$i^{***} = \frac{(1 - \theta)l + \frac{A}{\beta}}{1 - \beta \rho_0}$$

If $\beta(1+r_t) > 1$, then $x_t = 1$, and x_{t+1} and \dot{i}_t can be inferred from (15) and (13):

$$x_{t+1} = \frac{A(1+n) - \frac{b_t}{(1-\beta\rho_0)} - \beta\rho_0 \frac{l + \rho_0 \frac{i_{t-1}}{1+n}}{1-\beta\rho_0}}{\beta(1+n)\theta l} \quad \text{and} \quad \dot{i}_t = \frac{b_t + l + \rho_0 \frac{i_{t-1}}{1+n}}{1 - \beta \rho_0}$$

In order to trace the phase diagram, it is important to determine the relative positions of i^{**} and i^{***} . Let us show that it is always the case that $i^{***} > i^{**}$.

We have

$$i^{**} = \frac{A(1+n) + (1-\theta)l}{1 - \frac{\rho_0}{1+n}}$$

$$i^{***} = \frac{A\beta^{-1} + (1-\theta)l}{1 - \beta\rho_0}$$

Note that $\beta < \frac{1}{1+n}$. Consider the following function over the interval $[0, \frac{1}{1+n}]$

$$H(x) \equiv \frac{Ax^{-1} + (1 - \theta)l}{1 - x\rho_0}$$

We have $i^{***} = H(\beta)$ and $i^{**} = H(\frac{1}{1+n})$. Note that over this range, we have $1 - x\rho_0 > 0$ by assumption.

We can compute

$$\begin{aligned} H'(x) &= \frac{-Ax^{-2}(1 - x\rho_0) + \rho_0 [Ax^{-1} + (1 - \theta)l]}{(1 - x\rho_0)^2} \\ &= \frac{-Ax^{-2} + 2A\rho_0x^{-1} + \rho_0(1 - \theta)l}{(1 - x\rho_0)^2} \end{aligned}$$

Note that

$$\frac{d}{dx} [-Ax^{-2} + 2A\rho_0x^{-1} + \rho_0(1 - \theta)l] = 2Ax^{-3}(1 - \rho_0x) > 0$$

Hence for all $x \in [0, \frac{1}{1+n}]$, we have

$$-Ax^{-2} + 2A\rho_0x^{-1} + \rho_0(1 - \theta)l \leq -A(1 + n)^2 + 2A\rho_0(1 + n) + \rho_0(1 - \theta)l$$

We now use condition (B):

$$\frac{1 + n - 2\rho_0}{1 + n - \rho_0} \geq \frac{l}{(1 - \theta)l + A(1 + n)}$$

This implies the weaker condition

$$\frac{1 + n - 2\rho_0}{1 + n - \rho_0} \geq \frac{(1 - \theta)l}{(1 - \theta)l + A(1 + n)}$$

which we can rewrite as

$$(1 + n - 2\rho_0)A(1 + n) \geq \rho_0(1 - \theta)l$$

or equivalently

$$-A(1 + n)^2 + 2A\rho_0(1 + n) + \rho_0(1 - \theta)l \leq 0$$

This proves that for all $x \in [0, \frac{1}{1+n}]$, $H'(x) \leq 0$. Hence

$$i^{***} = H(\beta) > H\left(\frac{1}{1+n}\right) = i^{**}$$