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ABSTRACT

We create an analytical structure that reveals the long run risk-return relationship for nonlinear continuous time Markov environments. We do so by studying an eigenvalue problem associated with a positive eigenfunction for a conveniently chosen family of valuation operators. This family forms a semigroup whose members are indexed by the elapsed time between payoff and valuation dates. We represent the semigroup using a positive process with three components: an exponential term constructed from the eigenvalue, a martingale and a transient eigenfunction term. The eigenvalue encodes the risk adjustment, the martingale alters the probability measure to capture long run approximation, and the eigenfunction gives the long run dependence on the Markov state. We establish existence and uniqueness of the relevant eigenvalue and eigenfunction. By showing how changes in the stochastic growth components of cash flows induce changes in the corresponding eigenvalues and eigenfunctions, we reveal a long-run risk return tradeoff.

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1 Introduction

In financial economics risk-return tradeoffs show how expected rates of return over small intervals are altered as we change the exposure to the underlying shocks that impinge on the economy. In continuous time modeling, the length of interval is driven to zero to deduce a limiting local relationship. We study alternative notions of a risk-return relationship that feature the pricing of exposure to stochastic growth risk of cash flows. In contrast to the local analysis, we focus on what happens as the length of time between valuation and payoff becomes large.

The continuous time local analysis is facilitated by the use of stochastic differential equations driven by a vector Brownian motion and jumps. An equilibrium valuation model gives the prices of the instantaneous exposure of payoffs to these risks. Values over alternative horizons can be inferred by integrating appropriately these local prices. Such computations are nontrivial when there are nonlinearities in the evolution of state variables or valuations. This leads us to adopt an alternative approach based on an operator formulation of asset pricing. As in previous research, we model asset valuation using operators that assign prices today to payoffs in future dates. Since these operators are defined for each payoff date, we build an indexed family of such pricing operators. This family is referred to as a *semigroup* because of the manner in which the operators are related to one another.[1] We show how to modify valuation operators in a straightforward way to accommodate stochastic cash flow growth. It is the evolution of such operators as the payoff date changes that interests us. Long run counterparts to risk-return tradeoffs are reflected in how the limiting behavior of the family of operators changes as we alter the stochastic growth of the cash flows.

We study the evolution of the family of valuation operators in a continuous-time framework, although important aspects of our analysis are directly applicable to discrete-time specifications. Our analysis is made tractable by assuming the existence of a Markov state that summarizes the information in the economy pertinent for valuation. The operators we use apply to functions of this Markov state and can be represented as:

$$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x\right]$$

for some process M appropriately restricted to ensure intertemporal consistency and to

[1] See Garman (1984) for an initial contribution featuring the use of semigroups in modeling asset prices.

guarantee that the Markov structure applies to pricing. The principal restriction we impose is that M has a "multiplicative" property, resulting in a family of operators indexed by t that is a semigroup.

A central mathematical result that we establish and exploit is a multiplicative decomposition:

$$M_t = \exp(\rho t)\hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \tag{1}$$

where $\hat{M}$ is a martingale and its logarithm has stationary increments.[2] While such a representation is typically not unique, there is one such decomposition that is of value in our study of long term approximation. In this decomposition, ρ is a deterministic growth rate, and the ratio of positive random variables $\frac{\phi(x_0)}{\phi(x_t)}$ is a transitory contribution. In our analysis, we use the martingale $\{\hat{M}_t\}$ to change the probability measure prior to our study of approximation. The principal eigenfunction ϕ gives the dominant transient component of the operator family in the long run.

We use the multiplicative decomposition (1) to study two alternative long-run counterparts to risk-return tradeoffs. It allows us to isolate enduring components to cash-flows or prices and to explore the how these components are valued. For instance, cash flows with different stochastic growth components are valued differently. One long-run notion of a risk-return tradeoff characterizes how the asymptotic rate of return required as compensation depends on the cash flow risk exposure. A second approach cumulate returns that are valued in accordance to a local risk return tradeoff. A corresponding long-run tradeoff gives the asymptotic growth rates of alternative cumulative returns over long time horizons as a function of the risk exposures used to construct the local returns.

Previous papers have explored particular characterizations of the term structure of risk pricing of cash flows. (For instance, see Hansen et al. (2005) and Lettau and Wachter (2005).) In this regard local pricing characterizes one end of this term structure and our analysis the other end. Hansen et al. (2005) characterize the long-run cash flow risk prices for discrete time log-normal environments. Their characterization shares our goal of pricing the exposure to stochastic growth risk, but they exclude potential nonlinearity and temporal variation in volatility, in order to obtain analytical results. Hansen et al. (2005) examines the extent to which the long-run cash-flow risk prices from a family

[2]Alvarez and Jermann (2005) proposed the use of such a decomposition to analyze the long run behavior of stochastic discount factors and cited an early version of our paper for proposing the link between this decomposition and principal eigenvalues and functions. We develop this connection formally and establish existence and uniqueness results for a general class of multiplicative functionals.

of recursive utility models can account for the value heterogeneity observed in equity portfolios with alternative ratios of book value to market value. Our paper shows how to produce such pricing characterizations for more general nonlinear Markov environments.

The positive function ϕ and the scalar ρ are the principal eigenfunction and eigenvalue respectively of the semigroup. Specifically,

$$\mathbb{M}_t \phi(x) = \exp(\rho t)\phi(x). \tag{2}$$

so that ϕ is an eigenfunction of all of the operators $\mathbb{M}_t$ of the semigroup. There is a corresponding equation that holds locally, obtained essentially by differentiating with respect to t and evaluating the derivative at $t = 0$. More generally, this time derivative gives rise to the *generator* of the semigroup. By working with the generator, we exploit some of the well known local characterizations of continuous time Markov models from stochastic calculus to provide a solution to (2). While continuous-time models achieve simplicity by characterizing behavior over small time increments, operator methods have promise for enhancing our understanding of the connection between short run and long run behavior.

The remainder of the paper is organized as follows. In sections 2 and 3 we develop some of the mathematical preliminaries pertinent for our analysis. Specifically, in section 2 we give the underlying generation of the Markov process and introduce the reader to the concepts of additive and multiplicative functionals. Both functionals are crucial ingredients to what follows. In section 3 we introduce the reader to the notion of a semigroup. Semigroups are used to evaluate contingent claims written on the Markov state indexed by the elapsed time between trading date and the payoff date. In sections 4, 5 and 6 we consider three alternative multiplicative functionals that are pertinent in intertemporal asset pricing. In section 4 we use a multiplicative functional to model a stochastic discount factor process and the corresponding pricing semigroup. In section 5 we introduce valuation functionals that are used to represent returns over intervals of any horizon. In section 6 we introduce growth functionals to model nonstationary cash flows. Section 7 gives alternative notions of the generator of semigroup and discusses their relation. In section 8 we introduce principal eigenvalues and functions and use these objects to establish a representation of the form (1). In section 9 we establish formally the long run domination of the principal eigenfunction and eigenvalue and establish uniqueness of such objects for the purposes of approximation. In section 10 we discuss sufficient conditions for the existence of principal eigenvalues needed to support our analysis. Applications to financial economics are given in section 11.

2 Markov and related processes

We first describe the underlying Markov process and then we build other convenient processes from this underlying Markov process.

2.1 Baseline process

Let $\{X_t : t \geq 0\}$ be a continuous time, strong Markov process defined on a probability space $\{\Omega, \mathcal{F}, Pr\}$ with values on a state space $\mathcal{D}_0$ that is locally compact and separable. The sample paths of X are continuous from the right and with left limits, and we will sometimes also assume that this process is stationary and ergodic. Let $\mathcal{F}_t$ be completion of the sigma algebra generated by $\{X_u : 0 \leq u \leq t\}$.

Throughout we restrict the Markov process X to be a semimartingale. As a consequence, we can extract a continuous component X^c and what remains is a pure jump process X^j. To characterize the evolution of the jump component:

$$dX_t^j = \int_{\mathbb{R}^n} z\zeta(z, dt)$$

where $\zeta = \zeta(\omega, \cdot; \cdot)$ is a random counting measure. That is, for each ω, $\zeta(b, [0,t]; \omega)$ gives the total number of jumps in $[0,t]$ of a size in b, in the realization ω. In general, the associated Markov stochastic process X may have an infinite number of small jumps in any time interval. In what follows we will assume that this process has a finite number of jumps over a finite time interval. This rules out most Levy processes, but greatly simplifies the notation. In this case, there is a finite measure $\eta(dy|x)dt$ that is the *compensator* of the random measure ζ. It is the (unique) predictable random measure, such that for each predictable stochastic function $f(x, t; \omega)$, the process

$$\int_0^t \int_{\mathbb{R}^n} f(y, s; \omega)\zeta(dy, ds; \omega) - \int_0^t \int_{\mathbb{R}^n} f(y, s; \omega)\eta[dy|X_{s-}(\omega)]ds$$

is a martingale. The measure η encodes both a jump intensity and a distribution given that a jump occurs. The jump intensity is the implied conditional measure of the entire state space $\mathcal{D}_0$, and the jump distribution is the conditional measure divided by the jump intensity.

We presume that the continuous sample path component satisfies the stochastic evolution:

$$dX_t^c = \xi(X_{t-})dt + \Gamma(X_{t-})dB_t$$

where B is a multivariate $\mathcal{F}_t$-Brownian motion and $\Gamma(x)\Gamma(x)'$ is nonsingular. Given the rank condition, the Brownian increment can be deduced from the sample path of the state vector via:

$$dB_t = [\Gamma(X_{t-})'\Gamma(X_{t-})]^{-1}\Gamma(X_{t-})'[dX_t^c - \xi(X_{t-})dt].$$

Example 2.1. Finite State Markov Chain *Consider a finite state Markov chain with states* x_j *for* $j = 1, 2, ..., N$. *The local evolution of this chain is governed by an* $N \times N$ *intensity matrix* $\mathbb{U}$. *An intensity matrix encodes all of the transition probabilities. The matrix* $\exp(t\mathbb{U})$ *is the matrix of transition probabilities over an interval* t. *Since each row of a transition matrix sums to unity, each row of* $\mathbb{U}$ *sums to zero. The diagonal entries are negative and represent minus the intensity of jumping from the current state to a new one. The remaining row entries, appropriately scaled, represent the conditional probabilities of jumping to the respective states.*

Example 2.2. Markov Diffusion

In what follows we will often use the following example. Suppose the Markov process X *has two components,* X^f *and* X^o, *where* X^f *is a Feller square root process and is positive and* X^o *is an Ornstein-Uhlenbeck process and ranges over the real line:*

$$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f) + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o) + \sigma_o dB_t^o \end{aligned}$$

with $\xi_i > 0$, $\bar{x}_i > 0$ *for* $i = f, o$ *and* $2\xi_f\bar{x}_f \geq \sigma_f^2$ *where* $B = \begin{bmatrix} B^f \\ B^o \end{bmatrix}$ *is a bivariate standard Brownian motion. The parameter restrictions guarantee that there is a stationary distribution associated with* X^f *with support contained in* $\mathbb{R}_+$. [3]

[3]We could accommodate the case where B^f or B^o contain more than one entry, by considering a filtration $\{\mathcal{F}_t\}$ larger than the one generated by X. In effect, we would enlarge the state space in ways that were inconsequential to the computations that interest us. However, for simplicity we have assumed throughout this paper that $\{\mathcal{F}_t\}$ is the filtration generated by X.

Given an underlying Markov process X, we now explore ways of studying valuation and stochastic growth. This requires that we construct processes for stochastic discount factors and cash flow growth from the underlying Markov process. We do this by using building block processes that are additive and multiplicative functionals. In what follows we define formally a *functional*, an *additive functional* and a *multiplicative functional* and discuss their properties.

2.2 Additive functionals

A functional is a stochastic process constructed from the original Markov process:

Definition 2.1. *A real-valued process* $\{A_t : t \geq 0\}$ *is a* **functional** *if it is adapted (*A_t *is* $\mathcal{F}_t$ *measurable for all* t*) and is right continuous conditioned on* $X_0 = x$ *for almost everywhere for each* x *in* $\mathcal{D}_0$*.*

Let θ denote the shift operator. Using this notation, we define $A_u(\theta_t)$ to be the corresponding function of the X process shifted forward t time periods. Since A_u is constructed from the Markov process X between dates zero and u, $A_u(\theta_t)$ depends only on the process between dates t and date $t + u$.

Definition 2.2. *A functional* A *is* **additive** *if* $A_0 = 0$ *and* $A_{t+u} = A_u(\theta_t) + A_t$*, for each nonnegative* t *and* u*.*[4]

While the joint process $\{(X_t, A_t) : t \geq 0\}$ is Markov, by construction the additive functional does not *Granger cause* the original Markov process. Instead it is constructed from that process. No additional information about the future values of X are revealed by current and past value of A. When X is restricted to be stationary, an additive functional can be nonstationary but it has stationary increments. The following are examples of additive functionals:

Example 2.3. *Given any continuous function* ψ, $A_t = \psi(X_t) - \psi(X_0)$*.*

Example 2.4. *Let* β *be a Borel measurable function on* $\mathcal{D}_0$ *and construct:*

$$A_t = \int_0^t \beta(X_u) du$$

where $\int_0^t \beta(X_u) du < \infty$ *with probability one for each* t*.*

[4] Notice that we do not restrict additive functionals to have bounded variation as, *e.g.* Revuz and Yor (1994).

Example 2.5. *Form:*

$$A_t = \int_0^t \gamma(X_u)' dB_u$$

where $\int_0^t |\gamma(X_u)|^2 du$ *is finite with probability one for each* t.

Example 2.6. *Form:*

$$A_t = \sum_{0 \le u \le t} \kappa(X_u, X_{u-})$$

where $\kappa : \mathcal{D}_0 \times \mathcal{D}_0 \to \mathbb{R}$, $\kappa(x, x) = 0$.[5]

Sums of additive functionals are additive functionals. We may thus use examples 2.4, 2.5 and 2.6 as building blocks in a parameterization of additive functionals. This parameterization uses a triple (β, γ, κ) that satisfies:

a) $\beta : \mathcal{D}_0 \to \mathbb{R}$ and $\int_0^t \beta(X_u) du < \infty$ for every positive t;

b) $\gamma : \mathcal{D}_0 \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2 du < \infty$ for every positive t;

c) $\kappa : \mathcal{D}_0 \times \mathcal{D}_0 \to \mathbb{R}$, $\kappa(x, x) = 0$ for all $x \in \mathcal{D}_0$, $\int \exp(\kappa(y, x))\eta(dy|x) < \infty$ for all $x \in \mathcal{D}_0$.

Form:

$$\begin{aligned} A_t &= \int_0^t \beta(X_u) du + \int_0^t \gamma(X_{u-})' dB_u + \sum_{0 \le u \le t} \kappa(X_u, X_{u-}), \\ &= \int_0^t \beta(X_u) du + \int_0^t \gamma(X_{u-})' [\Gamma(X_{u-})' \Gamma(X_{u-})]^{-1} \Gamma(X_{u-})' [dX_u^c - \xi(X_{u-}) du] \\ &\quad + \sum_{0 \le u \le t} \kappa(X_u, X_{u-}). \end{aligned}$$

This additive functional is a semi-martingale.

While we will use extensively these parameterizations of an additive functional, they are not exhaustive as the following example illustrates.

Example 2.7. *Suppose that* $\{X_t : t \ge 0\}$ *is a standard scalar Brownian motion,* b *a Borelian in* $\mathbb{R}$, *and define the occupation time of* b *up to time* t *as*

$$A_t \doteq \int_0^t \mathbf{1}_{\{X_u \in b\}} du.$$

[5] Since the process has left limits, $X_{u-} = \lim_{t \uparrow u} X_t$ is well defined.

A_t is an additive functional. As a consequence, the local time at a point r defined as

$$L_t \doteq \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{X_u \in (r-\epsilon, r+\epsilon)\}} du,$$

is also an additive functional.

Multiplicative functionals can be built from additive functionals.

Definition 2.3. *The functional M is* **multiplicative** *if $M_0 = 1$, and $M_{t+u} = M_u(\theta_t)M_t$.*

Products of multiplicative functionals are multiplicative functionals. Exponentials of additive functionals are strictly positive multiplicative functionals. Conversely, the logarithm of strictly positive multiplicative functional is an additive functional.

3 Multiplicative functionals and semigroups

Given a multiplicative functional M, our aim is to establish properties of the family of operators:

$$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | x_0 = x\right]. \tag{3}$$

3.1 Semigroups

Let L be a Banach space with norm $\|\cdot\|$, and let $\{\mathbb{T}_t : t \geq 0\}$ be a family of operators on L. The operators in these family are linked according to the following property:

Definition 3.1. *A family of linear operators $\{\mathbb{T}_t : t \geq 0\}$ is a one-parameter* **semigroup** *if $\mathbb{T}_0 = \mathbb{I}$ and $\mathbb{T}_{t+s} = \mathbb{T}_t \mathbb{T}_s$ for all $s, t \geq 0$.*

One possibility is that these operators are conditional expectations operators, in which case this link typically follows from the Law of Iterated Expectations restricted to Markov processes. We will also use such families of operators to study valuation and pricing. From a pricing perspective, the semigroup property follows from the Markov version of the Law of Iterated Values, which holds when there is frictionless trading at intermediate dates.

We will often impose further restrictions on semigroups such as:

Definition 3.2. *The semigroup $\{\mathbb{T}_t : t \geq 0\}$ is* **positive** *if for any $t \geq 0$, $\mathbb{T}_t \psi \geq 0$ whenever $\psi \geq 0$.*

3.2 Multiplicative semigroup

Let $(L, \|\cdot\|)$ denote the Banach space of bounded functions of a Markov state under the sup norm. Among other things, the following assumption guarantees that (3) defines a bounded operator on L for every t.

Assumption 3.1. *The multiplicative functional M has finite first moments for each t.*

Proposition 3.1. *Suppose the multiplicative functional M satisfies Assumption 3.1. Then*

$$\mathbb{M}_t\psi(x) = E\left[M_t\psi(X_t)|X_0 = x\right].$$

is a semigroup on the space of bounded measurable functions.

Proof. Suppose that ψ is a bounded measurable function. Then

$$\begin{aligned}
\mathbb{M}_{t+u}\psi(x) &= E\left(E\left[M_{t+u}\psi(X_{t+u})|\mathcal{F}_t\right]|X_0 = x\right) \\
&= E\left[E\left(M_tM_u(\theta_t)\psi[X_u(\theta_t)]|\mathcal{F}_t\right)|X_0 = x\right] \\
&= E\left[M_tE\left[M_u(\theta_t)\psi[X_u(\theta_t)]|X_0(\theta_t)\right]|X_0 = x\right] \\
&= E\left[M_t\mathbb{M}_u\psi(x_t)|X_0 = x\right] \\
&= \mathbb{M}_t\mathbb{M}_u\psi(x),
\end{aligned}$$

which establishes the semigroup property. □

In what follows we will refer to semigroups constructed from multiplicative functionals as in this proposition as *multiplicative semigroups*.

Since the logarithm of a strictly positive multiplicative process is an additive process we will consider parameterized versions of strictly positive multiplicative processes by parameterizing the corresponding additive process. For instance, if $M = \exp(A)$ when A is parameterized by (β, γ, κ) satisfying the parameterization in section 2, we will say that the multiplicative process M is parameterized by (β, γ, κ). Notice that Ito's lemma guarantees that:

$$\frac{dM_t}{M_{t-}} = \left[\beta(X_{t-}) + \frac{|\gamma(X_{t-})|^2}{2}\right]dt + \gamma(X_{t-})'dB_t + \exp\left[\kappa(X_t, X_{t-})\right] - 1.$$

The multiplicative process $\{M_t : t \geq 0\}$ of this form is a local martingale if, and only if,

$$\beta + \frac{|\gamma|^2}{2} + \int \left(\exp\left[\kappa(y, \cdot)\right] - 1\right)\eta(dy|\cdot) = 0. \tag{4}$$

We next consider a variety of ways in which multiplicative functionals and their semigroups can be used when building models of asset prices and characterizing the resulting implications.

4 Pricing semigroup

For a fixed time interval t and any ψ in L, consider a state-contingent payoff $\psi(X_t)$. A pricing operator $\mathbb{S}_t$ applied to ψ gives the time zero price of this state-contingent payoff. This price is a function of the date zero state. We construct such an operator for every horizon t, giving us a family of operators that are naturally restricted to be a semigroup.

Formally, we build the pricing semigroup using a stochastic discount factor process $\{S_t : t \geq 0\}$ via:

$$\mathbb{S}_t\psi(x) = E\left[S_t\psi(X_t)|x_0 = x\right]. \tag{5}$$

The stochastic discount factor process is restricted to be a strictly positive multiplicative functional satisfying Assumption 3.1. With intermediate trading dates, the time $t + s$ payoff $\psi(X_{t+s})$ can be purchased at date zero or alternatively the claim could be purchased at date t at the price $\mathbb{S}_s\psi(X_t)$ and in turn this time t claim can be purchased at time zero. The semigroup property captures the notion that the date zero prices of $\psi(X_{t+s})$ and $\mathbb{S}_s\psi(X_t)$ must coincide. Thus the semigroup property is an iterated value property that connects pricing over different time intervals. It is a version of the *Law of One Price* as it applies over time in a Markov version of a frictionless market model with intermediate trading dates.[6]

Although we use the stochastic discount factor to price claims that are functions of the Markov state at a future date, once we choose a stochastic discount factor we can also price claims which are functions of the whole history up to a certain date.

Example 4.1. Breeden model

Using the Markov process given in example 2.2, we consider a special case of Breeden (1979)'s consumption-based asset pricing model. Suppose that equilibrium consumption evolves according to:

$$dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o. \tag{6}$$

[6]This semigroup property is related to the "consistency axiom" in Rogers (1998). In fact, adding the Markov property to the "axiomatic approach" of Rogers (1998) yields equation (5) with a multiplicative S.

where c_t is the logarithm of consumption C_t. Suppose also that investor's preferences are given by:

$$E\int_0^\infty \exp(-\mathsf{b}t)\frac{{C_t}^{1-\mathsf{a}}-1}{1-\mathsf{a}}$$

for a *and* b *strictly positive. The implied stochastic discount factor is $S_t = \exp(A_t^s)$ where*

$$A_t^s = -\mathsf{a}\int_0^t X_s^o ds - \mathsf{b}t - \mathsf{a}\int_0^t \sqrt{X_s^f}\vartheta_f dB_s^f - \mathsf{a}\int_0^t \vartheta_o dB_s^o.$$

Example 4.2. Kreps-Porteus model

As an alternative specification of preferences, suppose consumers have preferences that satisfy the recursion:

$$\lim_{\epsilon\downarrow 0}\frac{E\left(W_{t+\epsilon}-W_t|\mathcal{F}_t\right)}{\epsilon} = W_t\left[\mathsf{b}(\mathsf{a}-1)c_t + \mathsf{b}\log W_t\right]$$

where $-W_t$ is the continuation value for the consumption plan, $\mathsf{a} > 1$ *and* $\mathsf{b} > 0$*. These preferences can be viewed as a continuous time version of the preferences suggested by Kreps and Porteus (1978) and is a special case of the stochastic differential utility model of Duffie and Epstein (1992) and Schroder and Skiadas (1999). For these preferences the intertemporal composition of risk matters. Bansal and Yaron (2004) have used this feature of preferences in conjunction with predictable components in consumption and consumption volatility as a device to amplify risk premia. The particular utility recursion we use imposes a unitary elasticity of intertemporal substitution.*

Suppose again that consumption evolves according to equation (6). Conjecture a continuation value process of the form:

$$W_t = \exp\left[(1-\mathsf{a})(\mathsf{w}_f X_t^f + \mathsf{w}_o X_t^o + c_t + \bar{\mathsf{w}})\right]$$

The coefficients satisfy:

$$\begin{aligned}
-\xi_f\mathsf{w}_f + \frac{(1-\mathsf{a})\sigma_f^2}{2}(\mathsf{w}_f)^2 + (1-\mathsf{a})\vartheta_f\sigma_f\mathsf{w}_f + \frac{(1-\mathsf{a})\vartheta_f^2}{2} &= \mathsf{b}\mathsf{w}_f\\
-\xi_o\mathsf{w}_o + 1 &= \mathsf{b}\mathsf{w}_o\\
\xi_f\bar{x}^f\mathsf{w}_f + \xi_o\bar{x}^o\mathsf{w}_o + \frac{(1-\mathsf{a})\sigma_o^2}{2}(\mathsf{w}_o)^2 + (1-\mathsf{a})\vartheta_o\sigma_o\mathsf{w}_o + \frac{(1-\mathsf{a})\vartheta_o^2}{2} &= \mathsf{b}\bar{\mathsf{w}}.
\end{aligned}$$

The stochastic discount factor is the product of two multiplicative functionals. One has *the same form as the Breeden model with* $\mathsf{a} = 1$*. It is the exponential of:*

$$A_t^s = -\int_0^t X_s^o ds - \mathsf{b}t - \int_0^t \sqrt{X_s^f}\vartheta_f dB_s^f - \int_0^t \vartheta_o dB_s^o.$$

The other is a martingale. It is the contribution from the continuation value and is the exponential of:

$$\begin{aligned} A_t^w &= (1-\mathsf{a})\int_0^t \sqrt{X_s^f}(\vartheta_f + \mathsf{w}_f\sigma_f)dB_s^f + (1-\mathsf{a})\int_0^t (\vartheta_o + \mathsf{w}_o\sigma_o)dB_s^o \\ &\quad -\frac{(1-\mathsf{a})^2}{2}\int_0^t X_s^f \frac{|\vartheta_f + \mathsf{w}_f\sigma_f|^2}{2}ds - \frac{(1-\mathsf{a})^2}{2}t \end{aligned}$$

5 Valuation functionals and returns

A valuation process is constructed to have the following property. If the future value of the process is the payout, the current value is the price of that payout. For instance a valuation process could be the result of continually reinvesting dividends in a primitive asset. Equivalently, it can be constructed by continually compounding realized returns to an investment. To characterize local pricing, we use valuation processes that are multiplicative functionals. Recall that the product of two multiplicative functionals is a multiplicative functional. The following definition is motivated by the connection between the absence of arbitrage and the martingale properties of properly normalized prices.

Definition 5.1. *A* **valuation functional** $\{V_t : t \geq 0\}$ *is a multiplicative functional such that the product functional* $\{V_t S_t : t \geq 0\}$ *is a martingale.*

Provided that V is strictly positive, the associated gross returns over any horizon u can be calculated by forming ratios:

$$R_{t,t+u} = \frac{V_{t+u}}{V_{t-}}$$

Thus increments in the value functional scaled by the current value gives an instantaneous net return. The martingale property of the product VS gives a local pricing restriction for returns.

To deduce a convenient and familiar risk return relation, consider the multiplicative functional $M = VS$ where V is parameterized by $(\beta_v, \gamma_v, \kappa_v)$ and $\{S_t : t \geq 0\}$ is parameterized by $(\beta_s, \gamma_s, \kappa_s)$. In particular, the implied net return evolution is:

$$\frac{dV_t}{V_{t-}} = \left[\beta_v(X_{t-}) + \frac{|\gamma_v(X_{t-})|^2}{2}\right] dt + \gamma_v(X_{t-})' dB_t + \exp\left[\kappa_v(X_t, X_{t-})\right] - 1.$$

Thus the expected net rate of return is:

$$\varepsilon_v \doteq \beta_v + \frac{|\gamma_v|^2}{2} + \int \left(\exp\left[\kappa_v(y, \cdot)\right] - 1\right) \eta(dy, \cdot).$$

Since both V and S are exponentials of additive processes, their product is the exponential of an additive process and is parameterized by:

$$\begin{aligned}\beta &= \beta_v + \beta_s \\ \gamma &= \gamma_v + \gamma_s \\ \kappa &= \kappa_v + \kappa_s\end{aligned}$$

Proposition 5.1. *A valuation process parameterized by $(\beta_v, \gamma_v, \kappa_v)$ satisfies the pricing restriction:*

$$\beta_v + \beta_s = -\frac{|\gamma_v + \gamma_s|^2}{2} - \int \left(\exp\left[\kappa_v(y, \cdot) + \kappa_s(y, \cdot)\right] - 1\right) \eta(dy, \cdot). \tag{7}$$

Proof. This follows from the definition of a valuation functional and the martingale restriction (4). □

This restriction is local and determines the instantaneous risk-return relation. The parameters (γ_v, κ_v) determine the Brownian and jump risk exposure. The following corollary gives the required local mean rate of return:

Corollary 5.1. *The required mean rate of return for the risk exposure (γ_v, κ_v) is:*

$$\varepsilon_v = -\beta_s - \gamma_v \cdot \gamma_s - \frac{|\gamma_s|^2}{2} - \int \left(\exp\left[\kappa_v(y, \cdot) + \kappa_s(y, \cdot)\right] - \exp\left[\kappa_v(y, \cdot)\right]\right) \eta(dy, \cdot)$$

The vector $-\gamma_s$ contains the factor risk prices for the Brownian motion components. The function κ_s is used to price exposure to jump risk. Then ε_v is the required expected rate of return expressed as a function of the risk exposure. This local relation is familiar from the extensive literature on continuous-time asset pricing.

Example 5.1. Breeden example continued

Consider again the Markov diffusion example 2.2 with the stochastic discount factor given in example 4.1. This is a Markov version of Breeden's model. The local risk price for exposure to the vector of Brownian motion increments is:

$$\begin{bmatrix} \mathsf{a}\sqrt{x_f}\vartheta_f \\ \mathsf{a}\vartheta_o \end{bmatrix}$$

and the instantaneous risk free rate is:

$$\mathsf{b} + \mathsf{a}x_o - \frac{\mathsf{a}^2\left(x_f(\vartheta_f)^2 + (\vartheta_o)^2\right)}{2}.$$

Consider a family of valuation processes parameterized by (β, γ) *where:* $\gamma(x) = (\sqrt{x_f}\gamma_f, \gamma_o)$. *To satisfy the martingale restriction, we must have:*

$$\beta(x) = \mathsf{b} + \mathsf{a}x_o - \frac{1}{2}\left(x_f(\gamma_f - \mathsf{a}\vartheta_f)^2 + (\gamma_o - \mathsf{a}\vartheta_o)^2\right)$$

Example 5.2. Kreps-Porteus example continued

Consider again the Markov diffusion example 2.2 with the stochastic discount factor given in example 4.2. The local risk price for exposure to the vector of Brownian motion increments is:

$$\begin{bmatrix} \mathsf{a}\sqrt{x_f}\vartheta_f + (\mathsf{a}-1)\sqrt{x_f}\mathsf{w}_f\sigma_f \\ \mathsf{a}\vartheta_o + (\mathsf{a}-1)\mathsf{w}_o\sigma_o \end{bmatrix}$$

and the instantaneous risk free rate is:

$$\mathsf{b} + \mathsf{a}x_o - \frac{\mathsf{a}}{2}\left(x_f\vartheta_f^2 + \vartheta_o^2\right) - (\mathsf{a}-1)\mathsf{w}_f x_f\vartheta_f\sigma_f - (\mathsf{a}-1)\mathsf{w}_o\vartheta_o\sigma_o.$$

As we have seen, alternative valuation functionals reflect alternative risk exposures. The methods we will describe will allow us to characterize the behavior of expectations of valuation functionals over long horizons, including long-horizon returns. To accomplish this we will be led to study semigroup $\{\mathbb{V}_t : t \geq 0\}$ constructed using the multiplicative functional V. While measurement of long-horizon returns in log-linear environments has commanded much attention, operator methods can accommodate low frequency volatility movements as well. In what follows, however, we will describe other ways to characterize a long-term risk return tradeoff.

6 Stochastic growth

The pricing semigroup we have thus far constructed only assigns prices to payoffs of form $\psi(X_t)$. When X is stationary, this specification rules out stochastic growth. We now extend the analysis to include payoffs with stochastic growth components by introducing a reference stochastic growth process: $\{G_t : t \geq 0\}$ that is a positive multiplicative functional. We will eventually restrict this process further. Consider a cash flow that can be represented as

$$D_t = G_t \psi(X_t) D_0 \tag{8}$$

for some initial condition D_0 where G multiplicative functional. For simplicity we may think of $\psi(X)$ as the stationary component of the cash flow and G as the growth component. However, as we will illustrate, the covariance between components sometimes makes this interpretation problematic.

The fact that the product of multiplicative functionals is a multiplicative functional facilitates the construction of valuation operators designed for cash flow processes that grow stochastically over time. We study cash flows with a common growth component using the semigroup:

$$\mathbb{Q}_t \psi(x) = E\left[G_t S_t \psi(X_t) | X_0 = x\right]$$

instead of pricing semigroup $\{\mathbb{S}_t\}$ constructed previously. The date zero price assigned to D_t is $D_0 \mathbb{Q}_t \psi(X_0)$. More generally, the date τ price assigned to $D_{t+\tau}$ is $D_0 G_\tau \mathbb{Q}_t \psi(X_\tau)$. Thus the date τ price to payout ratio is:

$$\frac{D_0 G_\tau \mathbb{Q}_t \psi(X_\tau)}{D_\tau} = \frac{D_0 G_\tau \mathbb{Q}_t \psi(X_\tau)}{D_0 G_\tau \psi(X_\tau)} = \frac{\mathbb{Q}_t \psi(X_\tau)}{\psi(X_\tau)}.$$

This semigroup assigns values to cash flows with common growth component G but alternative transient contributions ψ. To study how valuation is altered when we change stochastic growth, we will be led to alter the semigroup.

When the growth process is degenerate and equal to unity, the semigroup is identical to the one constructed previously in section 4. This semigroup is useful in studying the valuation of stationary cash flows including discount bonds and the term structure of interest. It supports local pricing and generalizations of the analyses of Backus and Zin (1994) and Alvarez and Jermann (2005) that use fixed income securities to make inferences about economic fundamentals. In our investigation of long run valuation, the

object	multiplicative functional	semigroup
stochastic discount factor	S	$\{\mathbb{S}_t\}$
cumulated return	V	$\{\mathbb{V}_t\}$
stochastic growth	G	$\{\mathbb{G}_t\}$
valuation with stochastic growth	$Q = GS$	$\{\mathbb{Q}_t\}$

Table 1: Alternative Semigroups and Multiplicative Functionals

study of this particular semigroup corresponds to one with no long term risk exposure. Nevertheless, the study of this semigroup offers a convenient benchmark for the study of long term risk just as a risk free rate offers a convenient benchmark in local pricing.

The decomposition (8) used in this construction is not unique. For instance, let φ be a strictly positive function of the Markov state. Then

$$D_t = G_t \psi(X_t) D_0 = \left[G_t \frac{\varphi(X_t)}{\varphi(X_0)} \right] \left[\frac{\psi(X_t)}{\varphi(X_t)} \right] \left[D_0 \varphi(X_0) \right].$$

Since $\frac{\psi(X_t)}{\varphi(X_t)}$ is a transient component, we can produce (infinitely) many such decompositions. For decomposition (8) to be unique, we must thus restrict the growth component.

A convenient restriction is to require that $G_t = \exp(\delta t)\hat{G}_t$ where $\hat{G}$ is a martingale. With this choice, by construction G has a constant conditional growth rate δ. Later we show how to extract martingale components, $\hat{G}$'s, from a large class of multiplicative functionals G. In this way we will establish the existence of such a decomposition. Even with this restriction, the decomposition will not always be unique, but we will justify a particular choice.

As we have seen, semigroups used for valuing growth claims are constructed by forming products of two multiplicative functionals, a stochastic discount factor functional and a growth functional. Pricing stationary claims and constructing cumulative returns lead to the construction of alternative multiplicative functionals. Table 1 gives a reminder of the alternative multiplicative functionals and semigroups. For this reason, we will study the behavior of a general multiplicative semigroup:

$$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x \right]$$

for some strictly positive multiplicative functional M. An important vehicle in this study is the *generator* of the semigroup.

7 Generator of a Multiplicative Semigroup

In section 3 we defined a multiplicative semigroup $\{\mathbb{M}_t : t \geq 0\}$ on a Banach space L^∞ of bounded functions equipped with the sup-norm.

7.1 Strong continuity

Definition 7.1. *The semigroup* $\{\mathbb{T}_t : t \geq 0\}$ *is* **strongly continuous** *if for any* $\psi \in L$*:*

$$\lim_{t \downarrow 0} \|\mathbb{T}_t \psi - \psi\| = 0$$

for some Banach space L.

Strong continuity is known to imply an exponential bound on the growth of the semigroup:

$$\|\mathbb{T}_t\| \leq K \exp(\delta t).$$

for some $K \geq 1$ and some positive δ.

Strong continuity of the semigroup is also known to imply the existence of the generator $\mathbb{U}$ of the semigroup. This generator is a closed operator defined on a dense subset, $D(\mathbb{U})$, of L as:

$$\mathbb{U}\psi = \lim_{t \downarrow 0} \frac{\mathbb{T}_t \psi - \psi}{t}. \tag{9}$$

[See Ethier and Kurtz (1986) Corollary 1.6 on page 10.] The operator $\mathbb{U}$ is referred to as a generator because the semigroup may be constructed from $\mathbb{U}$. This construction uses the exponential formula:

$$\mathbb{T}_t = \exp(t\mathbb{U})$$

which is defined rigorously through the Yosida approximation.[7]

Under assumption 3.1, the multiplicative semigroup $\{\mathbb{M}_t : t \geq 0\}$ is well defined on L^∞. Under the following additional restriction, this semigroup is strongly continuous.

Assumption 7.1.

$$\lim_{t \downarrow 0} \sup_{x \in \mathcal{D}_0} E\left(|M_t - 1| \,|X_0 = x\right) = 0. \tag{10}$$

Example 7.1. Markov Chain Generator

[7]See Ethier and Kurtz (1986) page 12 for a formal construction.

Recall the finite state Markov chain example 2.1 with intensity matrix $\mathbb{U}$*. Let* u_{ij} *denote entry* (i, j) *of this matrix. Consider a multiplicative functional that is the product of two components. The first component decays at rate* β_i *when the Markov state is* x_i*. The second component only changes when the Markov process jumps from state* i *to state* j*, in which case the multiplicative functional is scaled by* $\exp[\kappa(\mathsf{x}_j, \mathsf{x}_i)]$*. From this construction we can deduce the generator* $\mathbb{A}$ *for the multiplicative semigroup depicted as a matrix with entry* (i, j)*:*

$$\mathsf{a}_{ij} = \begin{cases} \mathsf{u}_{ii} - \beta_i & \text{if} \quad i = j \\ \mathsf{u}_{ij} \exp[\kappa(\mathsf{x}_j, \mathsf{x}_i)] & \text{if} \quad i \neq j \end{cases}.$$

This formula uses the fact that in computing the generator we are scaling probabilities by the potential proportional changes in the multiplicative functional. The matrix $\mathbb{A}$ *is not necessarily an intensity matrix. The row sums are not necessarily zero. The reason for this is that the multiplicative functional can include pure discount effects or pure growth effects. These effects can be present even when the* β_i*'s are zero since it is typically the case that*

$$\sum_{j \neq i} \mathsf{u}_{ij} \exp[\kappa(\mathsf{x}_j, \mathsf{x}_i)] \neq -\mathsf{u}_{ii}.$$

Establishing strong continuity is difficult in many applications. Even when strong continuity can be verified, it is difficult to characterize the domain of the generator. Moreover, the space of bounded functions used as the domain of the multiplicative semigroup is, for many purposes, too small. It is advantageous to have more flexibility when studying the local behavior of the semigroup.

7.2 A More General Construction

We now consider a more general definition of a generator. One of the advantages of this extended notion is that it allows for the use of Ito's formula to compute the generator.

Definition 7.2. *A Borel function* ψ *belongs to the domain of the* **extended generator** $\mathbb{A}$ *of the multiplicative functional* M *if there exists a Borel function* χ *such that* $N_t = M_t\psi(X_t) - \psi(X_0) - \int_0^t M_s\chi(X_s)ds$ *is a local martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$*. In this case the extended generator assigns* χ *to* ψ*.*

For strictly positive multiplicative processes M the extended generator is single valued and linear. As we show in appendix A, if $\{\mathbb{M}_t : t \geq 0\}$ is the semigroup defined by equation (3) and ψ is in $D(\mathbb{U})$, the extended generator assigns $\mathbb{U}\psi$ to ψ, and thus it is actually an extension of the generator $\mathbb{U}$ of $\{\mathbb{M}_t : t \geq 0\}$.

In the remainder of the paper, if the context is clear, we often refer to the extended generator simply as the generator.

The unit function is a trivial example of a multiplicative functional. In this case the extended generator is exactly what is called in the literature the extended generator of the Markov process X. Ito's formula allows us to obtain a characterization of this extended generator for any C^2 function, including ones that are not bounded. The extended generator has a well known representation:

$$\mathbb{A}\phi(x) = \xi(x) \cdot \frac{\partial \phi(x)}{\partial x} + \frac{1}{2}\text{trace}\left(\Sigma(x)\frac{\partial^2 \phi(x)}{\partial x \partial x'}\right) + \int [\phi(y) - \phi(x)]\, \eta(dy|x). \tag{11}$$

where

a) jump measure: $\eta(dy|x)$, a positive finite Borel measure for each x;

b) first derivative term: an n-dimensional vector ξ of functions;

c) second derivative term: a (pointwise) positive, semidefinite matrix $\Sigma = \Gamma\Gamma'$ of functions .

Recall our earlier parameterization of an additive functional A in terms of the triple (β, γ, κ). The process $M = \exp(A)$ is a multiplicative functional. We now display how to go from the extended generator of the Markov process X, that is the generator associated with $M \equiv 1$, to the extended generator of the multiplicative functional M. Suppose that the conditional measure η is finite and the generator $\mathbb{A}$ of the Markov process is parameterized by (ξ, Σ, η) as in (11). The matrix Σ is restricted to have a constant rank on the interior of the state space.

The formulas below use the parameterization for the multiplicative process to transform the generator of the Markov process into the generator of the multiplicative semigroup and are consequences of Ito's lemma:

a) jump measure: $\exp[\kappa(y, x)]\, \eta(dy|x)$.

b) first derivative term: $\xi(x) + \Gamma(x)\gamma(x)$;

c) second derivative term: $\Sigma(x)$;

d) level term: $\beta(x) + \frac{|\gamma(x)|^2}{2} + \int (\exp[\kappa(y,x)] - 1)\,\eta(dy,x)$;

The Markov chain example that we discussed above can be seen as a special case where γ, ξ, and Σ are all null.

There are a variety of direct applications of this analysis. In the case of the stochastic discount factor introduced in section 4, the generator encodes the local prices reflected in the local risk return tradeoff of Proposition 5.1. The level term that arises gives the instantaneous version of a risk free rate. In the absence of jump risk, the increment to the drift gives the factor risk prices. The function κ shows us how to value jump risk in small increments in time.

In a further application, Anderson et al. (2003) use this decomposition to characterize the relation among four alternative semigroups, each of which is associated with an alternative multiplicative process. Anderson et al. (2003) feature models of robust decision making. In addition to the generator for the original Markov process, a second generator depicts the worst case Markov process used to support the robust equilibrium. There is a third generator of an equilibrium pricing semigroup, and a fourth generator of a semigroup used to measure the statistical discrepancy between the original model and the worst-case Markov model.

8 Principal eigenfunctions and martingales

As a precursor to our study of long run behavior, we construct what are called principal eigenfunctions of the generator of a multiplicative functional. We then use these eigenfunctions to obtain a decomposition of a multiplicative functional.

Definition 8.1. *A Borel function ϕ is an* **eigenfunction** *of the extended generator $\mathbb{A}$ with* **eigenvalue** *ρ if $\mathbb{A}\phi = \rho\phi$.*

Proposition 8.1. *Suppose that ϕ is an eigenfunction of the extended generator associated with the eigenvalue ρ. Then*

$$\exp(-\rho t) M_t \phi(X_t)$$

is a local martingale.

Proof. Since the local martingale N_t is continuous from the right with left limits it is a semimartingale (Protter (2005), Chapter 3, Corollary to Theorem 30) and hence $Y_t =$

$M_t\phi(X_t)$ is also a semimartingale. Since $dN_t = dY_t - \rho Y_{t-}$, integration by parts yields:

$$\exp(-\rho t)Y_t - Y_0 = -\int_0^t \rho \exp(-\rho s)Y_{s-}ds + \int_0^t \exp(-\rho s)dY_s = \int_0^t \exp(-\rho s)dN_s.$$

□

It is the strictly positive eigenfunctions that interest us.

Definition 8.2. *A* **principal eigenfunction** *of the extended generator is an eigenfunction that is strictly positive.*

Corollary 8.1. *Suppose that ϕ is a principal eigenfunction with eigenvalue ρ for the extended generator of the multiplicative functional M. Then this multiplicative functional can be decomposed as:*

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)}\right].$$

where $\hat{M}_t = \exp(-\rho t)M_t \frac{\phi(X_t)}{\phi(X_0)}$ is a local martingale that is a multiplicative functional.

Proof. Note that when ϕ is strictly positive, then

$$\hat{M}_t = \exp(-\rho t)M_t \frac{\phi(X_t)}{\phi(X_0)}$$

is a multiplicative functional. □

Let $\hat{M}$ be the local martingale from Corollary 8.1. Since $\hat{M}$ is bounded from below, the local martingale is necessarily a supermartingale and thus

$$E\left(\hat{M}_t|\mathcal{F}_u\right) \leq \hat{M}_u.$$

We are primarily interested in the case in which this local martingale is actually a martingale:

Assumption 8.1. *The local martingale $\hat{M}$ is a martingale.*

By examining the proof of Proposition 8.1, one verifies that a sufficient condition for Assumption 8.1 to hold is that the local martingale N is a martingale. In appendix B we give primitive conditions that imply Assumption 8.1.

When Assumption 8.1 holds we may define for each event $f \in \mathcal{F}_t$

$$\hat{P}r(f) = E[\hat{M}_t \mathbf{1}_f]$$

The probability $\hat{P}r$ is absolutely continuous with respect to Pr when restricted to $\mathcal{F}_t$ for each $t \geq 0$. In addition, if we write $\hat{E}$ for the expected value taken using $\hat{P}r$, we obtain:

$$E\left[M_t \psi(X_t) | X_0 = x\right] = \exp(\rho t)\phi(x)\hat{E}\left[\frac{\psi(X_t)}{\phi(X_t)} | X_0 = x\right] \tag{12}$$

If we treat $\exp(-\rho t)\phi(X_t)$ as a *numeraire*, equation (12) is reminiscent of the familiar risk-neutral pricing in finance. Note, however, that the numeraire depends on the eigenvalue-eigenfunction pair, and equation (12) applies even when the multiplicative process does not define a price.[8]

Although ϕ does not necessarily belong to the Banach space L where the semigroup $\{\mathbb{M}_t : t \geq 0\}$ was defined, under Assumption 8.1 we can always define $\mathbb{M}_t \phi$. In fact:

Proposition 8.2. *Under Assumption 8.1, for each $t \geq 0$*

$$\mathbb{M}_t \phi = \exp(\rho t)\phi$$

Proof.

$$1 = E[\hat{M}_t | X_0 = x] = \frac{\exp(-\rho t)}{\phi(x)} E[M_t \phi(X_t) | X_0 = x].$$

□

This proposition guarantees that under Assumption 8.1, a principal eigenvector of the extended generator also solves the principal eigenvalue problem:

$$\mathbb{M}_t \phi = \exp(\rho t)\phi. \tag{13}$$

On the other hand if $\phi \in L$ solves equation (13) for each $t \geq 0$, then $\mathbb{U}\phi = \rho\phi$ where $\mathbb{U}$ is the generator defined in (9). Since the extended generator extends $\mathbb{U}$, ϕ is a principal eigenvector of the extended generator.

In light of the decomposition given by Corollary 8.1, when the local martingale $\hat{M}$ is a martingale, we will sometimes refer to ρ as the growth rate of the multiplicative

[8] The idea of using an appropriately chosen eigenfunction of an operator to construct and analyze a distorted probability measure is also featured in the work of Balaji and Meyn (2000).

functional M, $\hat{M}$ as its martingale component and $\frac{\phi(X_0)}{\phi(X_t)}$ as its transient or stationary component. This decomposition is typically not unique, however. As we have defined them, there may be multiple principal eigenfunctions even after a scale normalization. Each of these principal eigenfunctions implies a distinct decomposition (provided that we establish that the associated local martingales are martingales.) Since the martingale and the stationary components are correlated, it can happen that $E\left[\hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} | X_0 = x\right]$ diverges exponentially challenging the interpretation that ρ is the asymptotic growth rate of the semigroup. We take up this issue in the next section.

Remark 8.1. *There are well known martingale decompositions of additive functionals with stationary increments used in deducing central limit approximation and in characterizing the role of permanent shocks in time series. The nonlinear, continuous time Markov version of such a decomposition is:*

$$A_t = \omega t + m_t - \upsilon(X_t) + \upsilon(X_0)$$

where $\{m_t : t \geq 0\}$ *is a martingale with stationary increments (see Bharttacharya (1982) and Hansen and Scheinkman (1995)). Exponentiating this decomposition is of a similar type to that given in Corollary 8.1 except that the exponential of a martingale is not a martingale. When the martingale increments are constant functions of Brownian increments, then exponential adjustment has simple consequences.*[9] *In particular, the exponential adjustment is offset by changing* ω. *With state dependent volatility in the martingale approximation, however, there is no longer a direct link between the additive and multiplicative decompositions. In this case the multiplicative decomposition of Corollary 8.1 is the one that is valuable for our purposes.*

Example 8.1. Markov chain example

Recall that for a finite state space, we can represent the Markov process in terms of a matrix $\mathbb{U}$ *that serves as its generator. Previously we constructed the corresponding generator* $\mathbb{A}$ *of the multiplicative semigroup. For this example, the generator is a matrix. A principal eigenvector is found by finding an eigenvector of* $\mathbb{A}$ *with strictly positive entries. Standard Perron-Frobenius theory implies that there is such an eigenvector, and it is unique up to scale.*

While there is uniqueness in the case of finite state chain, there can be multiple

[9]This is the case studied by Hansen et al. (2005).

solutions in more general settings.

Example 8.2. Markov diffusion example continued

Consider a multiplicative process $M = \exp(A)$ *where:*

$$A_t = \bar{\beta} t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o, \tag{14}$$

where X^f *and* X^o *are given in Example 2.2.*

Guess an eigenfunction of the form: $\exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$. *The corresponding eigenvalue equation is:*

$$\begin{aligned} \rho &= \bar{\beta} + \beta_f x_f + \beta_o x_o + \frac{\gamma_f^2}{2} x_f + \frac{\gamma_o^2}{2} \\ &\quad + \mathsf{c}_f[\xi_f(\bar{x}_f - x_f) + x_f \gamma_f \sigma_f] + \mathsf{c}_o[\xi_o(\bar{x}_o - x_o) + \gamma_o \sigma_o] \\ &+ (\mathsf{c}_f)^2 x_f \frac{\sigma_f^2}{2} + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2} \end{aligned}$$

This generates two equations: one that equates the coefficients of x_f *to zero and another that equates the coefficients of* x_o *to zero:*

$$\begin{aligned} 0 &= \beta_f + \frac{\gamma_f^2}{2} + \mathsf{c}_f(\gamma_f \cdot \sigma_f - \xi_f) + (\mathsf{c}_f)^2 \frac{\sigma_f^2}{2} \\ 0 &= \beta_o - \mathsf{c}_o \xi_o. \end{aligned}$$

The solution to the first equation is:

$$\mathsf{c}_f = \frac{(\xi_f - \gamma_f \sigma_f) \pm \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right)}}{|\sigma_f|^2} \tag{15}$$

provided that

$$(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right) \geq 0.$$

Later we will argue that only one of these solutions interests us. The solution to the second equation is:

$$\mathsf{c}_o = \frac{\beta_o}{\xi_o}. \tag{16}$$

The resulting eigenvalue is:

$$\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + \mathsf{c}_f \xi_f \bar{x}_f + \mathsf{c}_o(\xi_o \bar{x}_o + \gamma_o \sigma_o) + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2}.$$

Write

$$\hat{M}_t = \exp(-\lambda t) M_t \frac{\exp(\mathsf{c}_f X_t^f + \mathsf{c}_o X_t^o)}{\exp(\mathsf{c}_f X_0^f + \mathsf{c}_o X_0^o)},$$

Then $\hat{M}$ is a martingale. It may be verified that $\hat{M}_t = \exp(\hat{A}_t)$ where:

$$\hat{A}_t = \int_0^t \sqrt{X_s^f}(\gamma_f + \mathsf{c}_f\sigma_f) dB_s^f + \int_0^t (\gamma_o + \mathsf{c}_o\sigma_o) dB_s^o - \frac{(\gamma_f + \mathsf{c}_f\sigma_f)^2}{2} \int_0^t X_s^f ds - \frac{(\gamma_o + \mathsf{c}_o\sigma_o)^2}{2} t$$

This martingale appends a drift to each component of the Brownian motion. The resulting drift for X^f is

$$\xi_f(\bar{x}_f - x_f) + x_f\sigma_f(\gamma_f + \mathsf{c}_f\sigma_f),$$

and distorted drift for X^o is:

$$\xi_o(\bar{x}_o - x_o) + \sigma_o(\gamma_o + \mathsf{c}_o\sigma_f).$$

We pick a solution for c_f so that the implied distorted process for X^f remains stationary. Notice that

$$\xi_f(\bar{x}_f - x_f) + x_f\sigma_f(\gamma_f + \mathsf{c}_f\sigma_f) = \xi_f\bar{x}_f \pm x_f\sqrt{(\xi_f - \gamma_f\sigma_f)^2 - \sigma_f^2\left(2\beta_f + \gamma_f^2\right)}.$$

For mean reversion to exist, we require that the coefficient on x_f be negative.

Remark 8.2. *At a cost of an increase in notational complexity we could add an "affine" jump component as in Duffie et al. (2000). Suppose that the state variable X^o instead of being an Ornstein-Uhlenbeck process satisfies:*

$$dX_t^o = \xi_o(\bar{x}_o - X_t^o) + \sigma_o dB_t^o + dZ_t$$

where Z is a pure jump process whose jumps have a fixed probability distribution ν on $\mathbb{R}$ and arrive with intensity $\varpi_1 x_f + \varpi_2$ with $\varpi_1 \geq 0$, $\varpi_2 \geq 0$. Suppose that the additive functional A has an additional term jump term modeled using $\kappa(y,x) \doteq \bar{\kappa}(y_o - x_o)$ for $y \neq x$ and $\int \exp[\bar{\kappa}(z)] d\nu(z) < \infty$.

The generator $\mathbb{A}$ has now an extra term given by:

$$(\varpi_1 x_f + \varpi_2) \int [\phi(x_f, x_o + z) - \phi(x_f, x_o)] \exp[\bar{\kappa}(z)] d\nu(z).$$

Hence when $\phi(x) = \exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$ *the extra term reduces to:*

$$(\varpi_1 x_f + \varpi_2) \exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o) \int [\exp(\mathsf{c}_o z) - 1] \exp[\bar{\kappa}(z)] \, d\nu(z).$$

As before we must have

$$\mathsf{c}_o = \frac{\beta_o}{\xi_o}$$

and hence c_f *must solve:*

$$0 = \beta_f + \frac{\gamma_f^2}{2} + \mathsf{c}_f(\gamma_f \sigma_f - \xi_f) + (\mathsf{c}_f)^2 \frac{\sigma_f^2}{2} + \varpi_1 \int \left[\exp\left(\frac{\beta_o}{\xi_o} z \right) - 1 \right] \exp[\bar{\kappa}(z)] \, d\nu(z).$$

The resulting eigenvalue is

$$\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + \mathsf{c}_f \xi_f \bar{x}_f + \mathsf{c}_o(\xi_o \bar{x}_o + \gamma_o \sigma_o) + (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2} + \varpi_2 \int \left[\exp\left(\frac{\beta_o}{\xi_o} z \right) - 1 \right] \exp[\bar{\kappa}(z)] \, d\nu(z).$$

9 Long Run Dominance

We investigate long term risk by changing the reference growth functionals. These functionals capture the long term risk exposure of the cash flow. As we will demonstrate, the valuation of cash flows with common reference growth functionals will be approximated by a single dominant component when the valuation horizon becomes long. Thus the contributions to value that come many periods into the future will be approximated by a single pricing factor that incorporates an adjustment for risk. Changing the reference growth functional alters the long term risk exposure with a corresponding adjustment in valuation. Each reference growth functional will be associated with a distinct semigroup. We will characterize long term risk formally by studying the limiting behavior of the corresponding semigroup.

In this section we establish approximation results for the principal eigenvalue and eigenfunction as a device for the study of behavior of valuation of cash flows with payout dates far into the future and for the construction of asymptotic growth rates and discount rates of cash flows.

We first illustrate this dominance in the case of a Markov chain.

9.1 Markov chain

Consider again the finite state Markov example with intensity matrix $\mathbb{U}$. In this section we will study the long run behavior of the semigroup by solving the eigenvalue problem:

$$\mathbb{A}\phi = \rho\phi$$

for an eigenvector ϕ with strictly positive entries and a real eigenvalue ρ. Given this solution, then

$$\mathbb{M}_t\phi = \exp(t\mathbb{A})\,\phi = \exp(\rho t)\phi.$$

The beauty of Frobenius-Perron theory is that ρ is the eigenvalue that dominates in the long run. Its real part is strictly larger than the real parts of all of the other eigenvalues. This property dictates its dominant role. To see this, suppose that the matrix $\mathbb{A}$ has distinct eigenvalues:

$$\mathbb{A} = \mathbb{T}\mathbb{D}\mathbb{T}^{-1}.$$

where $\mathbb{T}$ is a matrix with eigenvectors in each column and $\mathbb{D}$ is a diagonal matrix of eigenvalues. Let the first entry of $\mathbb{D}$ be ρ and the first column of T beϕ. Then the first row of $\mathbb{T}^{-1}$ is the eigenvector ϕ^* associated with the principal eigenvalue ρ normalized so that $\phi^* \cdot \phi = 1$. Thus

$$\exp(t\mathbb{A}) = \mathbb{T}\exp(t\mathbb{D})\,\mathbb{T}^{-1}.$$

Scaling by $\exp(-\rho t)$ and taking limits:

$$\lim_{t\to\infty} \exp(-\rho t)\exp(t\mathbb{A})\,\psi = (\phi^* \cdot \psi)\phi.$$

Thus ρ determines the long-run growth rate of the semigroup. After adjusting for this growth, the semigroup has an approximate one factor structure in the long run. Provided that $\phi^* \cdot \psi$ is not zero, $\exp(-\rho t)\mathbb{M}_t\psi$ is proportional to the dominant eigenvector ϕ.

9.2 General analysis

To establish this dominance more generally, we use the same martingale construction as in the decomposition of Corollary 8.1 to build an alternative family of distorted Markov transition operators and apply known results about Markov operators to this alternative family.

In what follows we will maintain Assumption 8.1 and let $\hat{\mathbb{A}}$ denote the extended generator of the martingale $\hat{M}$. We will also call the semigroup $\hat{\mathbb{M}}$ associated with $\hat{M}$ the *principal eigenfunction semigroup.* This semigroup is well defined at least on the space L^{∞}, and it maps constant functions into constant functions.

Consistent with the applications that interest us, we consider only multiplicative functionals that are strictly positive.

Assumption 9.1. *The multiplicative functional M is strictly positive with probability one.*

We presume that there exists a stationary density for the process X under distorted evolution.

Assumption 9.2. *There exists a probability measure $\hat{\varsigma}$ such that*

$$\int \hat{\mathbb{A}}\psi d\hat{\varsigma} = 0$$

for all ψ in the L^{∞} domain of the generator $\hat{\mathbb{A}}$.

This second assumption implies that $\hat{\varsigma}$ is a stationary distribution for the distorted Markov process. (For example, see Proposition 9.2 of Ethier and Kurtz (1986).) In what follows we use the notation $\hat{E}$ and $\hat{\mathcal{P}r}$ to denote the expectation operator and the probability measure associated with $\hat{M}$ and $\hat{\varsigma}$. The process $\hat{M}$ determines the distorted transition probabilities and $\hat{\varsigma}$ is the initial distribution.

Let $\hat{\Delta} > 0$ and consider the discrete time Markov process obtained by sampling the process at $j\hat{\Delta}$ for $j = 0, 1, ...$ In what follows we assume that the resulting discrete time process is *irreducible.*

Assumption 9.3. *There exists a $\hat{\Delta} > 0$ such that the discretely sampled process $\{X_{\hat{\Delta}j} : j = 0, 1, ...\}$ is* **irreducible***. That is, for any Borel set Λ of the state space $\mathcal{D}_0$ with $\hat{\varsigma}(\Lambda) > 0$,*

$$\hat{E}\left[\sum_{j=0}^{\infty} \mathbf{1}_{\{X_{\hat{\Delta}j} \in \Lambda\}} | X_0 = x\right] > 0$$

for all $x \in \mathcal{D}_0$.

Under Assumption 9.1 it is equivalent to assume that this irreducibility restriction holds under the original probability measure.[10]

We establish approximation results by imposing a form of stochastic stability under the distorted probability measure. We assume that the distorted Markov process satisfies:

Assumption 9.4. *The process X is* **Harris recurrent** *under the measure $\hat{P}r$. That is, for any Borel set Λ of the state space $\mathcal{D}_0$ with positive $\hat{\varsigma}$ measure,*

$$\hat{P}r\left\{\int_0^\infty \mathbf{1}_{\{X_t\in\Lambda\}} = \infty | X_0 = x\right\} = 1$$

for all $x \in \mathcal{D}_0$.

Among other things, this assumption guarantees that the stationary distribution $\hat{\varsigma}$ is unique.

Under these assumptions, we characterize the role of the principal eigenvalue and function on the long run behavior of the semigroup.

Proposition 9.1. *Suppose that $\hat{M}$ satisfies Assumptions 8.1, 9.1 - 9.4, and let $\Delta > 0$.*

a. For any ψ for which $\int(|\psi|/\phi)d\hat{\varsigma} < +\infty$

$$\lim_{j\to\infty} \exp(-\rho\Delta j)\mathbb{M}_{\Delta j}\psi = \phi\int\frac{\psi}{\phi}d\hat{\varsigma}$$

for almost all ($\hat{\varsigma}$) x.

b. For any ψ for which ψ/ϕ is bounded,

$$\lim_{t\to\infty} \exp(-\rho t)\mathbb{M}_t\psi = \phi\int\frac{\psi}{\phi}d\hat{\varsigma}$$

for $x \in \mathcal{D}_0$.

Proof. Note that

$$\exp(-\rho t)\mathbb{M}_t\psi(x) = \hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right)\phi(x).$$

[10] Irreducibility and Harris recurrence are defined relative to a measure. This claim uses the $\hat{\varsigma}$ measure when verifying irreducibility for the original probability measure. Since irreducibility depends only on the probability distribution conditioned on X_0, it does not require that the X process be stationary under the original measure.

It follows from Theorem 6.1 of Meyn and Tweedie (1993a) that,

$$\lim_{t\to\infty} \sup_{0\leq\psi\leq\phi} \left| \hat{\mathbb{M}}_t \left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = 0,$$

which proves (b). Consider any sample interval $\Delta > 0$. Then

$$\lim_{j\to\infty} \sup_{0\leq\psi\leq\phi} \left| \hat{\mathbb{M}}_{\Delta j} \left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = 0.$$

From Proposition 6.3 of Nummelin (1984), the sampled process $\{X_{j\Delta} : j = 0, 1, ...\}$ is aperiodic and Harris recurrent with stationary density $\hat{\varsigma}$. Hence if $\int \left| \frac{\psi}{\phi}(x) \right| d\hat{\varsigma}(x) < \infty$,

$$\lim_{j\to\infty} \hat{\mathbb{M}}_{\Delta j} \left(\frac{\psi}{\phi}\right) = \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for almost all $(\hat{\varsigma})$ x, which proves (a). (See for example, Theorem 5.2 of Meyn and Tweedie (1992).)

□

The approximation implied by Proposition 9.1, among other things, gives a formal sense in which ρ is a long run growth rate. It also provides more precise information, namely that after eliminating the deterministic growth, application of the semigroup to ψ is approximately proportional to ϕ where the scale coefficient is $\int \frac{\psi}{\phi} d\hat{\varsigma}$. Subsequently, we will consider other versions of this approximation. We will also impose additional regularity conditions that will guarantee convergence without having to sample the Markov process.

9.2.1 Uniqueness

Recall that there may exist more than one principal eigenfunction of the extended generator even after a scale normalization is imposed. Stochastic stability requirements typically eliminate this multiplicity.

Example 9.1. *Consider again the Markov diffusion specification given in examples 2.2, 4.1, 4.2 and 8.2. We constructed two solutions to the generalized eigenvalue problem. Only one of these two solutions will be stochsatically stable. The other solution will result in Markov process that fails to be stationary. Recall that the two candidate drift* *distortions are:*

$$\xi_f \bar{x}_f \pm x_f \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 \left(2\beta_f + \gamma_f^2\right)}.$$

By selecting the solution associated with the negative root, we obtain a process with a stationary density.

It turns out that this approach to uniqueness works much more generally.

Proposition 9.2. *Suppose that Assumption 9.1 is satisfied and that there exists a sampling interval Δ such that $\{X_{j\Delta} : j = 0, 1, ...\}$ is irreducible. There is at most one (up to scale) principal eigenfunction ϕ of the extended generator $\mathbb{A}$ of a multiplicative process M for which the associated $\{\hat{M}_t : t \geq 0\}$ satisfies Assumptions 8.1, 9.2-9.4.*

Proof. Consider two such principal eigenfunctions, ϕ and ϕ^*, and let $\rho \geq \rho^*$ be the corresponding eigenvalues. By Proposition 8.2 each eigenfunction-eigenvalue pair must solve the principal eigenvalue problem:

$$\begin{aligned} \mathbb{M}_t \phi(x) &= \exp(\rho t)\phi(x) \\ \mathbb{M}_t \phi^*(x) &= \exp(\rho^* t)\phi^*(x). \end{aligned}$$

If $\hat{M}$ is the martingale associated with the eigenvector ϕ, then

$$E\left[\hat{M}_t \frac{\phi^*(X_t)}{\phi(X_t)} | X_0 = x\right] = \exp[(\rho^* - \rho)t] \frac{\phi^*(x)}{\phi(x)}.$$

Since the discrete-time sampled Markov process associated with $\hat{M}$ is Harris recurrent, aperiodic and has a unique stationary distribution, the left-hand side converges to:

$$\hat{E}\left[\frac{\phi^*(X_0)}{\phi(X_0)}\right]$$

for $t = \Delta j$ as the integer j tends to ∞. While the limit could be $+\infty$, it must be strictly positive, implying, since $\rho \geq \rho^*$, that $\rho^* = \rho$ and

$$\hat{E}\left[\frac{\phi^*(X_0)}{\phi(X_0)}\right] = \frac{\phi^*(x)}{\phi(x)}.$$

Hence the ratio of the two eigenfunctions is constant. □

9.2.2 L^p approximation

When there exists a stationary distribution, it follows from Nelson (1958) that the semigroup $\{\hat{\mathbb{M}}_t : t \geq 0\}$ can be extended to $\hat{L}^p$ for any $p \geq 1$ constructed using the measure $d\hat{\varsigma}$. The semigroup is a weak contraction. That is, for any $t \geq 0$,

$$\|\hat{\mathbb{M}}_t \psi\|_p \leq \|\psi\|_p$$

where $\| \cdot \|_p$ is the $\hat{L}^p$ norm.

Proposition 9.3. *Under Assumption 9.2, for $p \geq 0$*

$$\left[\int |\mathbb{M}_t \psi|^p \left(\frac{1}{\phi^p}\right) d\hat{\varsigma}\right]^{\frac{1}{p}} \leq \exp(\rho t) \left[\int |\psi|^p \left(\frac{1}{\phi^p}\right) d\hat{\varsigma}\right]^{\frac{1}{p}}$$

provided that $\int |\psi|^p \left(\frac{1}{\phi^p}\right) d\hat{\varsigma} < \infty$.

Proof. This follows from the weak contraction property established by Nelson (1958) together with the observation that

$$\exp(-\rho t) \left(\frac{1}{\phi}\right) \mathbb{M}_t \psi = \hat{\mathbb{M}}_t \left(\frac{\psi}{\phi}\right).$$

□

Remark 9.1. *This proposition establishes an approximation in an L^p space constructed using the transformed measure $\frac{1}{\phi^p} d\hat{\varsigma}$. Notice that ϕ itself is always in this space. In particular, we may view the semigroup $\{\mathbb{M}_t : t \geq 0\}$ as operating on this space.*

Proposition 9.3 shows that when the distorted Markov process constructed using the eigenfunction is stationary, ρ can be interpreted as an asymptotic growth rate of the multiplicative semigroup. The eigenfunction is used to characterize the space of functions over which the bound applies. We now produce a more refined approximation.

Let $\hat{Z}^p$ denote the set of Borel measurable functions ψ such that $\int \psi d\hat{\varsigma} = 0$ and $\int |\psi|^p d\hat{\varsigma} < \infty$. Suppose that

Assumption 9.5. *For any $\psi \in \hat{Z}^p$,*

$$\lim_{t \to \infty} \|\hat{\mathbb{M}}_t \psi\|_p = 0.$$

In the case of $p = 2$, Hansen and Scheinkman (1995) give sufficient conditions for Assumption 9.5 to be satisfied.[11]

Proposition 9.4. *Under Assumptions 9.2 and 9.5, for any* ψ *such that* $\int \left|\frac{\psi}{\phi}\right|^p d\hat{\varsigma} < \infty$*,*

$$\left[\int \left|\exp(-\rho t)\mathbb{M}_t\psi - \phi \int \frac{\psi}{\phi} d\hat{\varsigma}\right|^p \frac{1}{\phi^p} d\hat{\varsigma}\right]^{\frac{1}{p}} \leq \mathsf{c}\exp(-\eta t).$$

for some rate $\eta > 0$ *and positive constant* c*.*

Proof. Notice that

$$\left|\exp(-\rho t)\mathbb{M}_t\psi - \phi \int \frac{\psi}{\phi} d\hat{\varsigma}\right| = \phi \left|\hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\varsigma}\right|.$$

Moreover,

$$\int \phi^p \left|\hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\varsigma}\right|^p \left(\frac{1}{\phi^p}\right) d\hat{\varsigma} = \int \left|\hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\varsigma}\right|^p d\hat{\varsigma}.$$

Assumption 9.5 implies that the right-hand side converges to zero as t gets large. By the semigroup property, this convergence is necessarily exponentially fast. □

9.2.3 Liapunov functions

Meyn and Tweedie (1993a) establish, under an additional mild continuity condition, sufficient conditions for the assumptions in this section using a "Liapunov function" method. In this subsection we will assume:

Assumption 9.6. *The process* X *is a Feller process under the probability measure associated with* $\hat{M}$*.*[12]

We use Liapunov functions that are restricted to be norm-like.

Definition 9.1. *A continuous function* V *is called* **norm-like** *if the set* $\{x : V(x) \leq r\}$ *is precompact for each* $r > 0$*.*

[11] Assumpion 9.5 for $p = 2$ is equivalent to requiring that the distorted Markov process be *rho-mixing*.

[12] By a Feller process we presume that the implied conditional expectation operators map continuous functions on the one-point compactification of $\mathcal{D}$ into continuous functions. In fact, Meyn and Tweedie (1993b) permit more general processes. The restriction that the process be Feller implies that all compact subsets are what Meyn and Tweedie (1993b) refer to as "petite sets."

A norm-like function converges to $+\infty$ along any sequence $\{x_j\}$ that converges to ∞. We will consider here only norm-like functions V for which $\hat{\mathbb{A}}V$ is continuous.

A sufficient condition for the existence of stationary distribution (Assumption 9.2) and for Harris recurrence (Assumption 9.4) is that there exists a norm-like function V for which

$$\frac{\mathbb{A}(\phi V)}{\phi} - \rho V = \hat{\mathbb{A}}V \leq -1$$

outside a compact subset of the state space. (See Theorem 4.2 of Meyn and Tweedie (1993b).)

In subsection 9.2.2 we established L^p approximations results. The space $\hat{L}^p$ is largest for $p = 1$. It is of interest to ensure that the constant functions are in the corresponding domain for the semigroup $\{\mathbb{M}_t : t \geq 0\}$. This requires that $1/\phi$ have a finite first moment under the stationary distribution $\hat{\varsigma}$. A sufficient condition for this is the existence of a norm-like function V such that

$$\mathbb{A}(\phi V) - \rho\phi V = \phi\hat{\mathbb{A}}\left(V\right) \leq -\max\left\{1, \phi\right\}$$

for x outside a compact set. (Again see Theorem 4.2 of Meyn and Tweedie (1993b).)

Finally, Proposition 9.4 only applies when the process is *weakly dependent* under the stationary distribution.[13] By weakening the sense of approximation we can expand the range of applicability. Consider some function $\hat{\psi} \geq 1$. For any t, we use

$$\sup_{|\psi| \leq \hat{\psi}} \left| \hat{\mathbb{M}}_t \psi - \int \psi d\hat{\varsigma} \right|$$

for each x as a measure of approximation. When $\hat{\psi} = 1$ this is equivalent to what is called the total variation norm by viewing $\hat{\mathbb{M}}_t\psi$ and $\int \psi d\hat{\varsigma}$ applied to indicator functions as measures for each x. It follows from Meyn and Tweedie (1993b) Theorem 5.3 that if there exists a norm-like function V and a real number a such that

$$\begin{aligned} \frac{\mathbb{A}(\phi V)}{\phi} - \rho V &= \hat{\mathbb{A}}V \leq -\hat{\psi} \\ \frac{\mathbb{A}(\phi\hat{\psi})}{\phi} - \rho\hat{\psi} &= \hat{\mathbb{A}}\hat{\psi} \leq \mathsf{a}\hat{\psi} \end{aligned} \tag{17}$$

[13] In contrast, Proposition 9.1 applies more generally.

outside a compact set, then

$$\phi \lim_{t\to\infty} \sup_{|\psi|\leq \phi\hat{\psi}} \left| \hat{\mathbb{M}}_t \frac{\psi}{\phi} - \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = \lim_{t\to\infty} \sup_{|\psi|\leq \phi\hat{\psi}} \left| \exp(-\rho t)\mathbb{M}_t \psi - \phi \int \frac{\psi}{\phi} d\hat{\varsigma} \right| = 0.$$

Note that in inequality (17) the constant a can be positive. Hence this inequality only requires the existence of an upper bound on rate of growth of the conditional expectation of the function $\hat{\psi}$ under the distorted probability. While the approximation is uniform in functions dominated by $\phi\hat{\psi}$ it is pointwise in the Markov state x.

The approximation results obtained in this section have a variety of applications depending upon our choice of the multiplicative functional M. In these applications M is constructed using stochastic discount factor functionals, growth functionals or valuation functions. These applications are described in section 11.

10 Existence

Thus far we have proceeded constructively. That is, we suppose that we can find solutions to principal eigenvalue problem and then proceed to check the alternative solutions. We now explore when solutions exist to this problem. For this, we follow Kontoyiannis and Meyn (2003) and Kontoyiannis and Meyn (2005) by working with a weighted L^∞ space. Suppose that there exist norm-like functions $V \geq 1$ and $W \geq 1$ such that

Assumption 10.1.

$$\frac{\mathbb{A}V}{V} \leq -W + \mathsf{a}$$

for $x \in \mathcal{D}_0$ and some positive number a.

In addition we will assume a continuity restriction on the multiplicative process M that is implied by Assumption 7.1 given previously

Assumption 10.2. *There exists an $\epsilon > 0$ such that $\int \exp(-\mathsf{a}t)E(M_t|X_0 = x)dt \geq \epsilon$ for all x.*

An alternative sufficient condition for Assumption 10.2 is that there exists a lower bound on the level term of the extended generator associated with M.

Our main result in this section is:

Proposition 10.1. *Suppose that Assumptions 10.1 and 10.2 hold. Then there exists a positive eigenfunction in the space of all functions of the form ψV where ψ is bounded.*

Proof. Construct the multiplicative process:

$$M_t^* = \exp(-\mathsf{a}t) M_t \frac{V(X_t)}{V(X_0)}.$$

Then

$$N_t^* = M_t^* - 1 - \int_0^t M_u^* \left[\frac{\mathbb{A}V(X_u)}{V(X_u)} - \mathsf{a} \right] du$$

is a local martingale, as we now verify. Note that

$$N_t = M_t V(x_t) - V(x_0) - \int_0^t M_u \mathbb{A}V(x_u) du$$

is a local martingale. Thus $\frac{1}{V(x_0)} \int_0^t \exp(-\mathsf{a}u) dN_u$ is also a local martingale and,

$$\begin{aligned}
& \frac{1}{V(x_0)} \int_0^t \exp(-\mathsf{a}u) dN_u \\
= \ & \exp(-\mathsf{a}t) M_t \frac{V(x_t)}{V(x_0)} - 1 \\
& + \mathsf{a} \int_0^t \exp(-\mathsf{a}u) M_u \frac{V(x_u)}{V(x_0)} du \\
& - \int_0^t \exp(-\mathsf{a}u) M_u \frac{V(x_u)}{V(x_0)} \frac{\mathbb{A}V(x_u)}{V(x_u)} du = N_t^*
\end{aligned}$$

Since $\{N_t^*\}$ is a local martingale, Fatou's Lemma implies that

$$E\left(M_t^* | x_0 = x\right) + \int_0^t E\left[M_u^* W(x_u) | x_0 = x\right] du \leq 1.$$

Since this holds for any t,

$$\int_0^\infty \exp(-\mathsf{a}t) E\left[M_t \frac{V(X_t) W(X_t)}{V(X_0) W(X_0)} | X_0 = x \right] dt \leq \frac{1}{W(x)} \tag{18}$$

Use the multiplicative process:

$$\tilde{M}_t = M_t \frac{V(X_t) W(X_t)}{V(X_0) W(X_0)}.$$

to construct the operator:[14]

$$\mathbb{F}\psi \doteq \int_0^\infty \exp(-\mathsf{a}t) E\left[M_t \frac{V(X_t)W(X_t)}{V(X_0)W(X_0)} \psi(X_t) | X_0 = x \right] dt = \int_0^\infty \exp(-\mathsf{a}t) \tilde{\mathbb{M}}_t \psi(x) dt.$$

Inequality (18) establishes that this operator maps the unit ball in L^∞ into a compact subset. Assumption 10.2 implies that $\mathbb{F}\frac{1}{VW} \geq \epsilon \frac{1}{VW}$, and as a consequence the spectral radius of $\mathbb{F}$ is strictly positive. (see Krasnosel'skij et al. (1989) Lemma 9.1 on page 89.) The Krein-Rutman theorem (See Krein and Rutman (1948) or Bonsall (1963)) guarantees that there exists a non-negative eigenfunction ϕ:

$$\mathbb{F}\phi = \lambda\phi$$

with a positive eigenvalue λ. Assumptions 9.3 and 9.1 guarantee that ϕ is strictly positive. Moreover,

$$\lambda \tilde{\mathbb{M}}_t \phi(x) = \tilde{\mathbb{M}}_t \mathbb{F}\phi(x) = \int_0^\infty \exp(-\mathsf{a}s) \tilde{\mathbb{M}}_{t+s} \phi(x) ds,$$

where the right-side follows from Tonelli's Theorem. Hence

$$\begin{aligned} \lambda \tilde{\mathbb{M}}_t \phi(x) &= \exp(\mathsf{a}t) \mathbb{F}\phi(x) - \exp(\mathsf{a}t) \int_0^t \exp(-\mathsf{a}s) \tilde{\mathbb{M}}_s \phi(x) ds \\ &= \exp(\mathsf{a}t) \lambda \phi(x) - \exp(\mathsf{a}t) \int_0^t \exp(-\mathsf{a}s) \tilde{\mathbb{M}}_s \phi(x) ds \end{aligned}$$

For a fixed x, define the function of t:

$$g(t) = \exp(-\mathsf{a}t) \tilde{\mathbb{M}}_t \phi(x).$$

Then

$$\lambda g(t) = \lambda \phi - \int_0^t g(s) ds$$

and $g(0) = \phi(x)$. The unique solution to this integral equation is

$$g(t) = \exp\left(-\frac{t}{\lambda}\right) \phi(x).$$

We now show that ϕ is an eigenfunction of the extended generator for the multiplica-

[14] $\mathbb{F}$ is a special case of a resolvent operator.

tive functional $\{\tilde{M}\}$. Consider

$$\tilde{N}_t \doteq \tilde{M}_t \phi(X_t) - \phi(X_0) - \left(\mathsf{a} - \frac{1}{\lambda}\right) \int_0^t \tilde{M}_s \phi(X_s) ds.$$

Take expectations conditioned on $\mathcal{F}_u$ for $0 \leq u < t$,

$$\begin{aligned} E\left(\tilde{N}_t | \mathcal{F}_u\right) &= \exp\left[(t-u)\left(\mathsf{a} - \frac{1}{\lambda}\right)\right] \tilde{M}_u \phi(X_u) - \phi(X_0) - \left(\mathsf{a} - \frac{1}{\lambda}\right) \int_0^u \tilde{M}_s \phi(X_s) ds \\ &\quad - \left(\mathsf{a} - \frac{1}{\lambda}\right) \tilde{M}_u \phi(X_u) \int_u^t \exp\left[(s-u)\left(\mathsf{a} - \frac{1}{\lambda}\right)\right] ds \\ &= \tilde{M}_u \phi(X_u) - \phi(X_0) - \left(\mathsf{a} - \frac{1}{\lambda}\right) \int_0^u \tilde{M}_s \phi(X_s) ds \\ &= \tilde{N}_u, \end{aligned}$$

which proves that $\{\tilde{N}_t\}$ is a martingale.

Since ϕ satisfies:

$$E\left[\tilde{M}_t \phi(X_t) | X_0 = x\right] = \exp\left[t\left(\mathsf{a} - \frac{1}{\lambda}\right)\right] \phi(x),$$

$$E\left[M_t V(X_t) W(X_t) \phi(X_t) | X_0 = x\right] = \exp\left[t\left(\mathsf{a} - \frac{1}{\lambda}\right)\right] V(x) W(x) \phi(x).$$

Therefore, $V(x)W(x)\phi(x)$ is the eigenfunction of interest. Finally, ϕW is bounded because of inequality (18). □

Proposition 10.1 shows that a principal eigenfunction exits. Moreover, as is evident from the proof, the resulting $\hat{M}$ is martingale, not just a local martingale. Also, the proof gives us some further insights into the probability distribution implied by $\hat{M}$. Using the parametrization in the proof, represent the principal eigenfunction as $VW\phi$. Since W is a norm-like function and $W\phi$ is bounded, $\frac{1}{\phi}$ is a norm-like function. Let ρ be the corresponding principal eigenvalue of the extended generator. Rewrite equation (18) in the proof as:

$$\begin{aligned} \hat{E} & \left[\int_0^\infty \exp[-(\mathsf{a} - \rho)t] \left(\frac{1}{\phi(X_t)}\right) dt | X_0 = x\right] \\ &= \int_0^\infty \exp(-\mathsf{a}t) E\left[M_t \frac{V(X_t)W(X_t)}{V(x)W(x)\phi(X_0)} | X_0 = x\right] \\ &\leq \frac{1}{\phi(x)W(x)} \end{aligned}$$

The operator on the left is a *resolvent operator* applied to the norm-like function $\frac{1}{\phi}$. It is obtained by taking the Laplace transform of the semigroup. We may use the stability methods of Down et al. (1995) to establish the stationarity of X under the distorted distribution and the convergence of the conditional expectation operators. Thus the approximation method developed in section 9 is applicable using the eigenfuntion shown to exist by Proposition 10.1.[15]

11 Long-Term Risk

A familiar result from asset pricing is the characterization of the short term risk return tradeoff. This tradeoff reflects the compensation, expressed in terms of expected returns, from being exposed to risk in the short run. Continuous time models of financial markets are revealing because they give a sharp characterization of this tradeoff by looking at the instantaneous limits. Our construction of valuation functionals in section 5 reflects this tradeoff in a continuous time Markov environment. Formally, the tradeoff is given in Corollary 5.1. In this section we explore another extreme, the tradeoff pertinent for the long run.

In the study of dynamical systems, long run analysis gives an alternative characterization that reveals different features from the short run dynamics. For linear systems it is easy to move from the short run to the long run. Nonlinearity makes this transformation much less transparent. This is precisely why operator methods are of value. It has long been recognized that steady state analysis provides a useful characterization of a dynamical system. For Markov processes the counterpart to steady state analysis is the analysis of a stationary distribution. We are led to a related but distinct analysis for two reasons. First, we consider economic environments with stochastic growth. Second, our interest is in the behavior of valuation, including valuation of cash flows with long run risk exposure. These differences lead us to study stochastic steady distributions under alternative probability measures.

As we have seen, these considerations lead naturally to the study of multiplicative semigroups that display either growth in expectation or decay in value. The counterpart to steady state analysis is the analysis of the principal eigenvalues and eigenfunctions, the objects that characterize the long run behavior of multiplicative semigroups. We

[15]As in our discussion Liapunov functions in section 9, we are presuming that compact subsets of the state space are *petite*.

use appropriately chosen eigenvalues and eigenfunctions to change probability measures. Changing probability measures associated with positive martingales are used extensively in asset pricing. Our use of this tool is distinct from the previous literature because of its role in long run approximation.

We now explore three alternative applications.

11.1 Decomposition of stochastic discount factors ($M = S$)

Alvarez and Jermann (2005) characterize the long run behavior of stochastic discount factors. Their characterization is based on a multiplicative decomposition on a permanent and a transitory component (see their Proposition 1). Corollary 8.1 delivers this decomposition, which we write as:

$$S_t = \exp(\rho t)\hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \tag{19}$$

for some martingale $\hat{M}$. The eigenvalue ρ is typically negative. We illustrated that such a decomposition is not unique. For such a decomposition to be useful in long-run approximation, the probability measure implied by martingale $\hat{M}$ must imply that the process X remains stationary. Proposition 9.2 shows that only one such representation implies that the process X remains recurrent and stationary under the change of measure.

Decomposition (19) of a stochastic discount factor funcational shows how to extract a deterministic growth component and a martingale component from the stochastic discount factor functional. Long run behavior is dominated by these two components *vis a vis* a transient component. Hansen et al. (2005) show that the transient component can often include contributions from habit persistence or social externalities as modeled in the asset pricing literature. As featured by Alvarez and Jermann (2005), this stochastic discount factor decomposition can be used to approximate prices of long term discount bonds:

$$\exp(-\rho t)E\left(S_t|X_0 = x\right) \approx \hat{E}\left[\frac{1}{\phi(X_t)}\right]\phi(x)$$

Thus ϕ is the dominant pricing factor in the long run. This approximation result extends more generally to stationary cash flows as characterized by Proposition 9.1.

11.2 Changing valuation functionals ($M = V$)

Alternative valuation functionals imply alternative risk exposures and growth trajectories. For one version of a long term risk return frontier, we change the risk exposure of the valuation functional subject to pricing restriction (7). This gives a family of valuation functionals that are compatible with a single stochastic discount factor. We may then apply the decomposition in Corollary 8.1 restricted so that the distorted Markov process is stationary to find a corresponding growth rate associated with each of these valuation functionals. Thus alternative valuation functionals as parameterized by the triple $(\beta_v, \gamma_v, \kappa_v)$ and restricted by the pricing restriction of Proposition 5.1 imply return processes with different long-run growth rates. The principle eigenvalues of the corresponding semigroups give these rates. In effect the valuation functionals can be freely parameterized by their risk exposure pair (γ_v, κ_v) with β_v determined by the local pricing restriction. The vector γ_v gives the exposure to Brownian risk and κ_v the exposure to jump risk.

Thus a long run risk-return frontier is given by the mapping from the risk exposure pair (γ_v, κ_v) to the long run growth rate of the valuation process. The growth rate may be computed by solving an eigenvalue problem that exploits the underlying Markovian dynamics. This characterizations allows us to move beyond the log-linear, log-normal specification implicit in many studies of long-horizon returns. The dominant eigenvalue calculation allows for conditional heteroskedasticity with long run consequences and it allows jumps that might occur infrequently. The principal eigenfunction (along with the eigenvalue) can be used to construct the martingale component as in Corollary 8.1.

Example 11.1. Application to the Markov diffusion example

Recall that in Breeden's model and the Kreps-Porteus model, the implied stochastic discount factor is $S_t = \exp(A_t^s)$ *where*

$$A_t^s = \bar{\beta}^s t + \int_0^t \beta_f^s X_s^f ds + \int_0^t \beta_o^s X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f^s dB_s^f + \int_0^t \gamma_o^s dB_s^o \tag{20}$$

where the alternative models give rise to alternative interpretations of the parameters. To parameterize a valuation functional $V = \exp(A^v)$*, we construct*

$$A_t^v = \bar{\beta}^v t + \int_0^t \beta_f^v X_s^f ds + \int_0^t \beta_o^v X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f^v dB_s^f + \int_0^t \gamma_o^v dB_s^o$$

where

$$\bar{\beta}^v + \beta^v_f x_f + \beta^v_o x_o = -\bar{\beta}^s - \beta^s_f x_f - \beta^s_o x_s - \frac{x_f}{2}(\gamma^s_f + \gamma^v_f)^2 - \frac{1}{2}(\gamma^s_o + \gamma^v_o)^2.$$

This equation imposes the local risk-return relation and determines $\bar{\beta}^v$, β^v_f and β^v_o as a function of the stochastic discount factor parameters and the risk exposure parameters γ^v_f and γ^v_o.

To infer the growth rates of valuation processes parameterized by (γ^v_f, γ^v_o), we find the principal eigenvalue for the multiplicative semigroup formed by setting $M = V$. Applying the calculation in Example 8.2, this eigenvalue is given by

$$\rho^v = \bar{\beta}^v + \frac{(\gamma^v_o)^2}{2} + \mathsf{c}^v_f \xi_f \bar{x}_f + \mathsf{c}^v_o(\xi_o \bar{x}_o + \gamma^v_o \sigma_o) + (\mathsf{c}^v_o)^2 \frac{\sigma_o^2}{2}.$$

where c^v_f and c^v_o are given by formulas (15) and (16) respectively. The terms on the right-hand side exclusive of $\mathsf{c}^v_f \xi_f \bar{x}_f$ give the continuous time log-normal adjustments, while $\mathsf{c}^v_f \xi_f \bar{x}_f$ adjusts for the stochastic volatility in the cumulative return. A long-run risk return tradeoff is given by mapping of (γ^v_f, γ^v_o) into the eigenvalue ρ^v.

11.3 Changing cash flows ($M = G$,$M = GV$)

Consider next a risky cash flow of the form:

$$D_t = G_t \psi(X_t) D_0$$

where G is a multiplicative functional. This cash flow grows over time. We could parameterize the multiplicative functional as the triple $(\beta_g, \gamma_g, \kappa_g)$, but this over-parameterizes the long-term risk exposure. The transient components to cash flows will not alter the long run risk calculation. One attractive possibility is to apply Corollary 8.1 and Propositions 9.1 and 9.2 with $(M = G)$ and use the martingale from that decomposition for our choice of G. Thus we could impose the following restriction on the G parameterization:

$$\beta_g + \frac{|\gamma_g|^2}{2} + \int (\exp[\kappa_g(y, \cdot)] - 1)\, \eta(dy, \cdot) = \delta$$

for some positive growth rate δ. Given δ this relation determines a unique β_g. In addition we restrict these parameters so that the distorted probability measure associated with an extended generator built from:

a) jump measure: $\exp\left[\kappa_g(y,x)\right]\eta(dy|x)$.

b) first derivative term: $\xi(x) + \Gamma(x)\gamma_g(x)$;

c) second derivative term: $\Sigma(x)$;

imply a semigroup conditional expectation operators that converge to the corresponding unconditional expectation operator.

Hansen et al. (2005) explore the valuation consequences by constructing a semigroup using $M = GS$ where S is a stochastic discount factor functional. They only consider the log-linear/log-normal model, however. Provided that we can apply Proposition 9.1 for this choice of M and ψ, the negative of the eigenvalue $-\rho$ is the overall rate of decay in value of the cash flow.

Consider an equity with cash flow D. For appropriate specifications of ψ, the values of the cash flows far into the future are approximately proportional to the eigenfunction ϕ. Thus we may view $\frac{1}{-\rho}$ as the limiting contribution to the price dividend ratio. The decay rate ρ reflects both a growth rate effect and discount rate effect. To net out the growth rate effect, we compute $-\rho + \delta$ as an asymptotic rate of return that encodes a risk adjustment. Heuristically, this is linked to Gordon growth model because $-\rho$ is the difference between the asymptotic rate of return $-\rho + \delta$ and the growth rate δ.

Following Hansen et al. (2005), we explore the consequences of altering the cash flow risk exposure. Such alterations induce changes in the asymptotic decay rate in value $(-\rho)$ and hence in the long-run dividend price ratio $\frac{1}{-\rho}$ and the asymptotic rate of return $-\rho + \delta$. The long-run cash flow risk return relation is captured by the mapping from the cash flow risk exposure pair (γ_g, κ_g) to the corresponding required rate of return $-\rho + \delta$.

Hansen et al. (2005) use this apparatus to produce such a tradeoff using empirical inputs in a discrete-time, log linear environment. The formulation developed here allows for extensions to include nonlinearity in conditional means, heteroskedasticity that contributes to long run risk and large shocks modeled as jump risk.

Example 11.2. Application to the Markov diffusion example

Returning to the Breeden model or the Kreps-Porteus model, suppose the growth process G is the exponential of the additive functional:

$$A_t^g = \delta t + \int_0^t \sqrt{X_s^f}\gamma_f^g dB_s^f + \int_0^t \gamma_o^g dB_s^o - \int_0^t \frac{X_s^f(\gamma_f^g)^2 + (\gamma_o^g)^2}{2} ds.$$

The parameters γ_f^g and γ_f^o parameterize the cash flow risk exposure. We limit the cash flow risk exposure by the inequality:

$$2(\xi_f + \sigma_f \gamma_f^g)\bar{x}_f \geq \sigma_f^2.$$

This limits the martingale component so that it induces stationarity under the probability measure induced by the $\hat{M}$ associated with $M = G$.

We use again the parameterization $S_t = \exp(A_t^s)$ where A_t^s is given by (20). Hence $A = A^s + A^g$ is given by

$$A_t = \bar{\beta} t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o$$

here $\bar{\beta} = \delta - \frac{(\gamma_o^g)^2}{2} + \bar{\beta}^s$, $\beta_f = \frac{(\gamma_f^g)^2}{2} + \beta_f^s$, $\beta_o = \beta_o^s$, $\gamma_f = \gamma_f^g + \gamma_f^s$ and $\gamma_o = \gamma_o^g + \gamma_o^s$. The formulas given in Example 8.2 discussed previously give us an asymptotic, risk adjusted rate of return,

$$-\rho + \delta = -\frac{(\gamma_o^s)^2}{2} - \gamma_o^s \gamma_o^g - \bar{\beta}^s - \mathsf{c}_f \xi_f \bar{x}_f - \mathsf{c}_o[\xi_o \bar{x}_o + (\gamma_o^g + \gamma_o^s)\sigma_o] - (\mathsf{c}_o)^2 \frac{\sigma_o^2}{2}.$$

Recall that $\mathsf{c}_o = \beta_o^s / \xi_o$ and c_f is a solution to a quadratic equation (15). This allows us to map exposures to the risks B^o and B^f into asymptotic rates of return. For instance, the long run risk prices to the cash flow risk exposure to the B^o risk is:

$$-\gamma_o^s - \frac{\beta_o^s}{\xi_o} \sigma_o$$

Because volatility is state dependent, the cash flow risk exposure to B^f is encoded in the coefficient c_f of the eigenfunction. Since this coefficient depends on γ_f^g in a nonlinear manner, there is nonlinearity in the long-run price of the volatility risk.

Hansen (2006) gives some other continuous time examples motivated by the work of Hansen et al. (2005) and Lettau and Wachter (2005) on cash flow risk and duration.

Hansen et al. (2005) also decompose the one period return risk to equity into a portfolio of one-period holding period returns to cash flows. Consider a cash flow of the form:

$$D_t = D_0 G_t \psi(X_t)$$

where G is a multiplicative growth functional. The limiting gross return is given by:

$$\lim_{t\to\infty} \frac{E\left(S_t D_t / S_1 | \mathcal{F}_1\right)}{E(S_t D_t | \mathcal{F}_0)} = \exp(-\rho) G_1 \frac{\phi(X_1)}{\phi(X_0)}$$

where ρ and ϕ are the principal eigenvalue and eigenfunction of the semigroup constructed using $M = GS$. This limit presumes that $\hat{E}\psi(X_t)$ is positive. The limiting holding period return has a cash flow growth component G_1, an eigenvalue component $\exp(-\rho)$ and an eigenfunction component $\frac{\phi(X_1)}{\phi(X_0)}$. The limit is independent of the transient contribution to the cash flow provided that assumptions of Proposition 9.1 are satisfied.

12 Conclusions

In this paper we characterized the long run risk-return relationship for nonlinear continuous time Markov environments. This long term relationship shows how alternative cash flow risk exposures are encoded in asymptotic risk-adjusted discount rates. To achieve this characterization we decompose a multiplicative functional built from the Markov process into the product of three components: (i) a deterministic exponential trend, (ii) a martingale whose logarithms has stationary increments and (iii) a transitory component. The transitory component is constructed from the principal eigenfunction evaluated of the semigroup associated with the multiplicative functional, and the rate of growth of the exponential trend is given by the corresponding eigenvalue. Hence we were led to study Frobenius-Perron notions of domination of semigroup.

There are several natural extensions of this work. First, while we have established existence and uniqueness of principal eigenvalues and eigenfunctions, it remains important to develop methods for computing these objects. There are only a limited array of examples for which quasi analytical solutions are currently available. We suspect that the weighted spaces we used in section 10 to establish existence will be helpful for computation. Second, while we have focused on dominant eigenvalues, more refined characterizations are needed to understand how well long run approximation works and how it can be improved. Results in Chen et al. (2005) could be extended and applied to achieve more refined characterizations, at least the case of multivariate diffusions. Third we considered only processes with a finite number of jumps in any finite interval of time. Extending this Frobenius-Perron theory to more general Levy processes may add new insights into characterizing long-term risk.

A Full generator

For a given multiplicative functional M, let $\tilde{L}$ be the space of all real valued functions ψ such that for every $t \geq 0$,

$$\sup_{0 \leq s < t} E[M_s |\psi(X_s)|] < \infty.$$

Consider a semigroup

$$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x\right].$$

Heuristically, we wish to compute a derivative with respect to time

$$\chi(x) = \lim_{t \downarrow 0} \frac{\mathbb{M}_t \psi(x) - \psi(x)}{t}.$$

Formally, we use the operator counterpart of the relation between an integral and its derivative to define what is referred to as the full generator.

Definition A.1. *The* **full generator** *of the semigroup* $\{\mathbb{M}_t : t \geq 0\}$ *is the subset of functions*

$$\overline{\mathbb{A}} = \left\{ (\psi, \chi) : \mathbb{M}_t \psi - \psi = \int_0^t \mathbb{M}_s \chi ds \right\}.$$

This notion of a generator is an extension of the generator given in section 7. Specifically, if ψ is in $D(\mathbb{U})$, the pair $(\psi, \mathbb{U}\psi)$ is in the full generator $\overline{\mathbb{A}}$. (See Davies (1980) Proposition 1.2.) The next Proposition establishes that the extended generator $\mathbb{A}$ extends the full generator.

Proposition A.1. *If* $(\psi, \chi) \in \overline{\mathbb{A}}$ *then* $N_t = M_t \psi(X_t) - \psi(X_0) - \int_0^t M_s \chi(X_s) ds$ *is a martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$.

Proof.

$$\begin{aligned} E[M_{t+u}\psi(X_{t+u})|\mathcal{F}_t] - E\left[\int_0^{t+u} M_s \chi(X_s) ds | \mathcal{F}_t\right] &= \\ M_t E\left[M_u(\theta_t)\psi(X_{t+u})|\mathcal{F}_t\right] - \int_0^t M_s \chi(X_s) ds - E\left[\int_t^{t+u} M_t M_{s-t}(\theta_t) \chi(X_s) ds | \mathcal{F}_t\right] &= \\ M_t \left[\mathbb{M}_u \psi(X_t) - \int_0^u \mathbb{M}_s \chi(X_t) ds\right] - \int_0^t M_s \chi(X_s) ds = N_t + \psi(X_0), \end{aligned}$$

since $(\psi, \chi) \in \overline{\mathbb{A}}$. □

.

B Martingales and Absolute Continuity

In this appendix we state some conditions that insure that Assumption 8.1 holds. Our result is inspired by the approach developed in chapter 7 of Liptser and Shiryaev (2000). Let $\hat{M}$ denote a multiplicative functional parameterized by $(\hat{\beta}, \hat{\gamma}, \hat{\kappa})$ that is restricted to be a local martingale. Thus $\hat{M} = \exp(\hat{A})$ where

$$\hat{A}_t = \int_0^t \hat{\beta}(X_u)du + \int_0^t \hat{\gamma}(X_{u-})'[\Gamma(X_{u-})'\Gamma(X_{u-})]^{-1}\Gamma(X_{u-})'[dX_u^c - \xi(X_{u-})du] + \sum_{0\leq u\leq t} \hat{\kappa}(X_u, X_{u-}),$$

and

Assumption B.1.

$$-\hat{\beta} - \frac{|\hat{\gamma}|^2}{2} - \int (\exp[\hat{\kappa}(y,x)] - 1)\,\eta(dy|x) = 0.$$

The extended generator for $\hat{M}$ is given by:

$$\begin{aligned}\hat{\mathbb{A}}\phi(x) = [\xi(x) + \Gamma(x)\hat{\gamma}(x)] \cdot \frac{\partial \phi(x)}{\partial x} \quad &+ \quad \frac{1}{2}\text{trace}\left(\Sigma(x)\frac{\partial^2\phi(x)}{\partial x \partial x'}\right) \\ &+ \quad \int [\phi(y) - \phi(x)]\exp[\hat{\kappa}(y,x)]\eta(dy|x).\end{aligned}$$

Assumption B.2. *There exists a probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{P}r)$, a filtration $\check{\mathcal{F}}_t$, an n-dimensional $\check{\mathcal{F}}_t$ Brownian motion $\check{B}$ and a semi-martingale $\check{X} = \check{X}^c + \check{X}^j$, where*

$$d\check{X}_t^c = [\xi(\check{X}_{t-}) + \Gamma(\check{X}_{t-})\hat{\gamma}(\check{X}_{t-})]dt + \Gamma(\check{X}_{t-})d\check{B}_t \tag{21}$$

and $\check{X}^j$ is a pure jump process with a finite number of jumps in any finite interval that has a compensator $\exp[\hat{\kappa}(y, \check{X}_{t-})]\eta(dy|\check{X}_{t-})dt$

In this case,

$$d\check{B}_t = [\Gamma(\check{X}_{u-})'\Gamma(\check{X}_{u-})]^{-1}\Gamma(\check{X}_{u-})'\left[d\check{X}_u^c - \xi(\check{X}_{u-})du - \Gamma(\check{X}_{u-})\hat{\gamma}(\check{X}_{u-})du\right].$$

Use the process $\check{X}$ to construct a multiplicative functional $\check{M} = \exp(\check{A})$ where

$$\check{A}_t \quad = \quad -\int_0^t \hat{\beta}(\check{X}_u)du - \int_0^t \hat{\gamma}(\check{X}_{u-})'[\Gamma(\check{X}_{u-})'\Gamma(\check{X}_{u-})]^{-1}\Gamma(\check{X}_{u-})'[d\check{X}_u^c - \xi(\check{X}_{u-})du]$$

$$
\begin{aligned}
& - \sum_{0 \le u \le t} \hat{\kappa}(\check{X}_u, \check{X}_{u-}) \\
= & - \int_0^t \left[\hat{\beta}(\check{X}_u) + |\gamma(\check{X}_{u-})|^2 \right] du \\
& - \int_0^t \hat{\gamma}(\check{X}_{u-})' [\Gamma(\check{X}_{u-})' \Gamma(\check{X}_{u-})]^{-1} \Gamma(\check{X}_{u-})' [d\check{X}_u^c - \xi(\check{X}_{u-}) du - \Gamma(\check{X}_{u-}) \hat{\gamma}(\check{X}_{u-}) du] \\
& - \sum_{0 \le u \le t} \hat{\kappa}(\check{X}_u, \check{X}_{u-}).
\end{aligned}
$$

The multiplicative functional $\check{M}$ is parameterized by:

$$
\begin{aligned}
\check{\beta} &= -\hat{\beta} - |\hat{\gamma}|^2 \\
\check{\gamma} &= -\hat{\gamma} \\
\check{\kappa} &= -\hat{\kappa}.
\end{aligned}
$$

Assumption B.3. *The parameterization* $(\check{\beta}, \check{\gamma}, \check{\kappa})$ *of the multiplicative functional* $\check{M}$ *satisfies:*

a) $\int_0^t \check{\beta}(\check{X}_u) du < \infty$ *for every positive* t*;*

b) $\int_0^t |\check{\gamma}(\check{X}_u)|^2 du < \infty$ *for every positive* t*.*

Notice that

$$
\int \exp[\check{\kappa}(y, x)] \hat{\eta}(dy|x) = \int \eta(dy|x) < \infty
$$

for all $x \in \mathcal{D}_0$. Moreover,

$$
\check{\beta} + \frac{|\check{\gamma}|^2}{2} + \int \left(\exp[\check{\kappa}(y, x)] - 1 \right) \hat{\eta}(dy|x) = -\hat{\beta} - \frac{|\hat{\gamma}|^2}{2} - \int \left(\exp[\hat{\kappa}(y, x)] - 1 \right) \eta(dy|x) = 0.
$$

Thus the multiplicative functional $\check{M}$ is a local martingale.

Proposition B.1. *Suppose that assumptions B.1, B.2 and B.3 are satisfied. Then the local martingale* $\hat{M}$ *is a martingale.*

Proof. We show that $\hat{M}$ is a martingale in three steps:

i) Since $\check{M}$ is a local martingale, there is an increasing sequence of stopping times $\{\check{\tau}_N : N = 1, ...\}$ that converge to ∞ such that

$$
\check{M}_t^N = \begin{cases} \check{M}_t & t \le \check{\tau}_N \\ \check{M}_{\check{\tau}_N} & t > \check{\tau}_N \end{cases}
$$

is a martingale and

$$\check{E}(\check{M}_t^N|\check{X}_0 = x) = 1$$

for all $t \geq 0$.

ii) Next we obtain an alternative formula for $\check{E}\left(\mathbf{1}_{\{t\leq\check{\tau}_N\}}|X_0 = x\right)$ represented in terms of the original X process. The stopping time $\check{\tau}_N$ can be represented as a function of $\check{X}$. Let τ_N be the corresponding function of X, and construct:

$$\hat{M}_t^N = \begin{cases} \hat{M}_t & t \leq \tau_N \\ \hat{M}_{\tau_N} & t > \tau_N \end{cases}.$$

Recall that

$$\hat{M}_t^N = \Phi_t(X)$$

for some Borel measurable function Φ_t. By construction,

$$\Phi_t(\check{X}) = \frac{1}{\check{M}_t^N}$$

Then

$$\begin{aligned} E\left(\hat{M}_t\mathbf{1}_{\{t\leq\tau_N\}}|X_0 = x\right) &= E\left(\hat{M}_t^N\mathbf{1}_{\{t\leq\tau_N\}}|X_0 = x\right) \\ &= \check{E}\left[\check{M}_t^N\left(\frac{1}{\check{M}_t^N}\right)\mathbf{1}_{\{t\leq\check{\tau}_N\}}|\check{X}_0 = x\right] \\ &= \check{E}\left(\mathbf{1}_{\{t\leq\check{\tau}_N\}}|\check{X}_0 = x\right) \end{aligned}$$

where the second equality follows from the Girsanov Theorem.

iii) Note that

$$\lim_{N\to\infty}\check{E}(\mathbf{1}_{\{\check{\tau}_N\leq t\}}|\check{X}_0 = x) = 1$$

by the Dominated Convergence Theorem. Thus

$$E(\hat{M}_t|X_0 = x) \geq \lim_{N\to\infty} E(\hat{M}_t\mathbf{1}_{\{\tau_N\leq t\}}|X_0) = \lim_{N\to\infty}\check{E}(\mathbf{1}_{\{\check{\tau}_N\leq t\}}|\check{X}_0 = x) = 1.$$

Since $\hat{M}$ is a nonnegative local martingale, we know that

$$E(\hat{M}_t|X_0 = x) \leq 1.$$

Therefore $E(\hat{M}_t|X_0 = x) = 1$ for all $t \geq 0$ and $\hat{M}$ is a martingale.

$\square$

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